$SU(3) \times SU(3)$ compactification and mirror duals of magnetic fluxes

Mariana Graña,[a] **Jan Louis**[b] **and Daniel Waldram**[c]

[a]*Service de Physique Théorique, CEA/ Saclay*
91191 Gif-sur-Yvette Cedex, France

[b]*II. Institut für Theoretische Physik der Universität Hamburg*
Luruper Chaussee 149, D-22761 Hamburg, Germany, and
Zentrum für Mathematische Physik, Universität Hamburg
Bundesstrasse 55, D-20146 Hamburg, Germany

[c]*Daniel Waldram*
Blackett Laboratory, Imperial College London
London, SW7 2BZ, U.K., and
Institute for Mathematical Sciences, Imperial College London
London, SW7 2PG, U.K.
E-mail: `mariana.grana@cea.fr`, `jan.louis@desy.de`, `d.waldram@imperial.ac.uk`

ABSTRACT: This paper analyses type II string theories in backgrounds which admit an $SU(3) \times SU(3)$ structure. Such backgrounds are designed to linearly realize eight out of the original 32 supercharges and as a consequence the low-energy effective action can be written in terms of couplings which are closely related to the couplings of four-dimensional $N = 2$ theories. This generalizes the previously studied case of $SU(3)$ backgrounds in that the left- and right-moving sector each have a different globally defined spinor. Given a truncation to a finite number of modes, these backgrounds lead to a conventional four-dimensional low-energy effective theory. The results are manifestly mirror symmetric and give terms corresponding to the mirror dual couplings of Calabi-Yau compactifications with magnetic fluxes. It is argued, however, that generically such backgrounds are non-geometric and hence the supergravity analysis is not strictly valid. Remarkably, the naive generalization of the geometrical expressions nonetheless appears to give the correct low-energy effective theory.

Contents

1. Introduction

String backgrounds which include non-trivial fluxes and are described by generalized geometry have been of considerable interest recently [1]. The primary reason is that such generalized compactifications are necessary whenever the string background contains D-branes . Generalized geometries have also featured prominently in recent mathematical investigations since they provide interesting extensions of certain established geometrical concepts such as complex and symplectic geometry [2]–[10].

A particular aspect of generalized geometries is that they can appear as mirror partners of Calabi-Yau compactifications with background fluxes [11]–[16] or as non-perturbative

duals of heterotic flux compactifications [17]. More specifically, if one considers type IIB supergravity compactified on Calabi-Yau threefolds one can turn on non-trivial three-form flux for both the Ramond-Ramond (RR) three-form F_3 and the Neveu-Schwarz (NS) three-form H_3. In the mirror symmetric type IIA background the RR three-form flux is mapped to RR-flux of the even field strength $F^+ = F_0 + F_2 + F_4 + F_6$ [18, 19]. On the other hand the NS three-form flux becomes part of the geometry in the mirror dual compactification [20, 11]. More precisely, a Calabi-Yau compactification with electric NS three-form flux is conjectured to be mirror symmetric to compactifications on manifolds known as "half-flat manifolds" [3, 7, 11].[1] These six-dimensional manifolds are a specific subclass of manifolds with $SU(3)$ structure. A generic manifold with $SU(3)$ structure admits a nowhere vanishing, globally defined spinor η which, however, is not necessarily covariantly constant with respect to the Levi-Civita connection. In this sense manifolds with $SU(3)$ structure generalize the notion of Calabi-Yau manifolds.[2]

The mirror of Calabi-Yau compactifications with magnetic three-form fluxes turns out to be more involved. The types of gaugings arising in such compactifications were discussed in [22]. In refs. [23 – 25] it has been suggested that the corresponding mirror backgrounds do not correspond to conventional geometric compactifications. Such non-geometrical backgrounds have been studied from different points of view in refs. [23]–[44]. In ref. [45] we conjectured that the mirror of the magnetic fluxes is found among compactifications on manifolds with $SU(3) \times SU(3)$ structure [8, 46, 47]. Such manifolds are generalizations of manifolds with $SU(3)$ structure in that they admit two globally defined spinors, one for each of the two original ten-dimensional supersymmetries. Recently the relationship between these different proposals has been clarified in ref. [42]. For $N = 1$ orientifold compactification our proposal for mirror symmetry was indeed confirmed in ref. [48]. Mirror symmetry can also be discussed in terms of brane configurations, which in this context are naturally described by calibrations in generalised geometry [8, 49].

In ref. [45] we showed that compactifications on manifolds with $SU(3) \times SU(3)$ structure are the most general geometric compactifications of type II supergravities with eight unbroken supercharges or, from a four-dimensional point of view, with $N = 2$ supersymmetry. The corresponding low-energy effective action depends only on the light modes of the string while the heavy string- and Kaluza-Klein excitations are integrated out. The couplings of this action are strongly constrained by the unbroken $N = 2$ supersymmetry which leads to an intricate interplay between supersymmetry and geometry. For generalized compactifications the distinction between heavy and light modes is not straightforward and as a consequence the definition of the effective action is somewhat ambiguous. In [45] we showed that even without any Kaluza-Klein truncation it is possible to rewrite the ten-dimensional effective action in a background with $SU(3)$-structure in a form which linearly

---

[1]The notion of electric flux is related to the definition of the Abelian (electric) gauge bosons. In type IIB they arise from expanding the RR four-form C_4 in terms of elements of the third cohomology H^3 of the Calabi-Yau. On H^3 there is a natural symplectic structure which in physical terms can be used to define electric gauge bosons and their magnetic duals. With this definition in mind one also has a natural split of the NS three-form flux into electric and magnetic. (See [19, 11] for further details.)

[2]In the context of string theory such manifolds were first discussed in [21].

realizes the eight unbroken supersymmetries. Or in other words we defined the equivalent of the standard $N = 2$ couplings, that is the holomorphic prepotential and the Killing prepotentials, but now in ten dimensions and showed that they do obey the constraints of $N = 2$ supersymmetry. From a four-dimensional point of view this action contains an infinite number of modes and a Kaluza-Klein reduction then corresponds to a consistent truncation to a finite subspace.

The purpose of this paper is to fill in two missing elements of our earlier work. We first reanalyze part of the reformulation of ten-dimensional type II supergravity in terms of Hitchin's generalized geometrical structures given in [45]. Specifically we derive the form of the Killing prepotentials (the $N = 2$ analogue of the superpotential and D-terms) in the case of a generic $SU(3) \times SU(3)$ structure, verifying the expressions conjectured in [45]. We then discuss the truncation to a finite set of modes, leading to a conventional four-dimensional effective theory. In this paper we do not address directly the question of when such truncations exist, but simply derive a set of consistency conditions for the effective theory to be $N = 2$ supersymmetric. (These issues are discussed in detail in [50].) Given such a truncation, we identify the backgrounds mirror to a Calabi-Yau compactification with magnetic H-flux, the case which was missing from the analysis of [11]. We then use existing work to argue that generically these are in fact non-geometrical. Nonetheless, the corresponding low-energy effective theories can be derived from the general $SU(3) \times SU(3)$ structure expressions, given some suitable truncation, despite the fact that these were derived assuming there was a geometrical compactification. This is consistent with the fact that at least some of the non-geometrical backgrounds are geometrical on any local patch.

The structure of the paper is as follows. In section 2 we review the geometry of generalized structures and show how they can be used to rewrite type II sypergravity in a form analogous to $d = 4$, $N = 2$ supergravity. In section 3, we show in detail how the spectrum of the supergravity fluctuations can be arranged into $N = 2$ — like multiplets and in addition, what representations need to be projected out in order to define a theory without additional spin-$\frac{3}{2}$ multiplets. In section 4 we derive the analogs of the Killing prepotentials for the generic theory, verifying the form conjectured in [45]. In section 5 we show that one can identify a specific $SU(3) \times SU(3)$ structure with an appropriate mode expansion of the supergravity fields which reproduces the mirror dual low-energy effective theory of Calabi-Yau compactifications with magnetic H-flux. In section 6 we consider generic $SU(3) \times SU(3)$ structures and compute the Killing prepotentials of the corresponding compactified type IIA and type IIB theories. They turn out to be manifestly mirror symmetric and all known compactifications can be obtained from them as special cases.[3] In section 7 we take up the issue of non-geometric compactifications and show that backgrounds with $SU(3) \times SU(3)$ structure generically also contain non-geometric backgrounds. Finally, section 8 concludes with some open problems. Our conventions for $Spin(6)$ and $Spin(6,6)$ spinors are given in appendix A while the conditions for a consistent mode truncation are spelled out in appendix B.

---

[3] A specific set of generalized mirror manifolds has been constructed in [51].

2. Supergravity and $SU(3) \times SU(3)$ structures

We begin by briefly reviewing the reformulation of ten-dimensional type II supergravities given in [45] and some of the key ingredients of generalized geometry in six dimensions. Recall that supersymmetry variations in type II supergravity are given by a pair of ten-dimensional spinors (ϵ^1, ϵ^2). In the reformulation, we concentrate on an eight-dimensional subset of supersymmetries, analogous to the eight supersymmetries of $N = 2$ supergravity in four $(d = 4)$ space-time dimensions. Since there are no eight-dimensional representations of $Spin(9, 1)$, this rewriting necessarily no longer has manifest ten-dimensional Lorentz symmetry, but, as we will see, the bosonic fields can actually be arranged in terms of $O(6, 6)$ representations which are the natural objects describing generalized geometry.

Specifically, decomposing $Spin(9, 1)$ into $Spin(3, 1) \times Spin(6)$ subgroups we identify eight supersymmetry parameters given by

$$
\begin{aligned}
\epsilon^1 &= \varepsilon^1_+ \otimes \eta^1_- + \varepsilon^1_- \otimes \eta^1_+ \,, \\
\epsilon^2 &= \varepsilon^2_+ \otimes \eta^2_\pm + \varepsilon^2_- \otimes \eta^2_\mp \,,
\end{aligned}
\tag{2.1}
$$

where in the second line we take the upper sign for type IIA and the lower for type IIB. Here η^A_+ with $A = 1, 2$ are spinors of $Spin(6)$ while ε^A are Weyl spinors of $Spin(3, 1)$. In each case η^A_- and ε^A_- are the charge conjugate spinors and the $\pm$ subscripts denote the chirality (for more details see appendix A). For a given pair (η^1_+, η^2_+) we have eight spinors parametrized by $\varepsilon^A_\pm$. These are the eight supersymmetries which remain manifest in the reformulated theory.

The assumption that we can identify η^A_+ globally puts a topological constraint on the ten-dimensional spacetime: it must admit a pair of $SU(3)$ structures, one for each spinor. The tangent bundle must split according to $TM^{9,1} = T^{3,1} \oplus F$, where F admits a pair of nowhere vanishing spinors. A simple example of such a split is a space-time which is a product $M^{9,1} = M^{3,1} \times M^6$ (with M^6 admitting two such spinors) but the background under consideration can also be more general. The split of the tangent space implies that all fields of the theory can be decomposed under $Spin(3, 1) \times Spin(6)$.

The two spinors η^A_+ are not necessarily different. If they coincide on the whole manifold, the two $SU(3)$ structures are the same, and the manifold has a single $SU(3)$ structure. In neighborhoods where the spinors are not parallel, two real vectors v and v' can be defined by the bilinear $v^m - iv'^m := \bar\eta^1_+ \gamma^m \eta^2_-$. If the spinors never coincide, this complex vector is nowhere vanishing, and the two $SU(3)$ structures intersect globally in an $SU(2)$ structure.

Instead of defining a general $SU(3)$ structure via the spinor η one can equivalently define it by a real fundamental two-form J and a complex three-form Ω. Analogously, a pair of $SU(3)$ structures can be defined by a pair (J^A, Ω^A) which locally (in neighborhoods where the two structures do not coincide) can be given as [52]

$$
\begin{aligned}
J^1 &= j + v \wedge v' \,, & \Omega^1 &= \omega \wedge (v + iv') \,, \\
J^2 &= j - v \wedge v' \,, & \Omega^2 &= \omega \wedge (v - iv') \,.
\end{aligned}
\tag{2.2}
$$

v, v' are one-forms, j is a real two-form and ω is a complex two-form. Together (j, ω, v, v') define a local $SU(2)$ structure on F, if none of them has zeros they define a global $SU(2)$ structure.

Crucially, one finds, following Hitchin [4, 8, 46], that the pair of $SU(3)$ structures is actually better viewed as an $SU(3) \times SU(3)$ structure on the generalized tangent bundle, that is $F \oplus F^*$. In turn, this structure is defined by a pair of $O(6,6)$ spinors. As a consequence, the bosonic supergravity fields can then all be written in terms of $O(6,6)$ representations. To briefly see how this works, let us start by recalling some facts about generalized geometry in six dimensions.

There is a natural $O(6,6)$ metric on $F \oplus F^*$ given by

$$(V, V') = i_x \xi' + i_{x'} \xi. \tag{2.3}$$

where $V = x + \xi, V' = x' + \xi' \in F \oplus F^*$. In a coordinate basis the metric reads

$$\mathcal{G} = \begin{pmatrix} 0 & \mathbb{I}_6 \\ \mathbb{I}_6 & 0 \end{pmatrix} . \tag{2.4}$$

Given this metric one can define $O(6,6)$ spinors. These are discussed in detail in the appendix A, here we will summarize some key points. It turns out that the spinor bundle S is isomorphic to the bundle of forms

$$S \simeq \Lambda^* F^* . \tag{2.5}$$

Spinors of $O(6,6)$ can be chosen to be Majorana –Weyl. The positive and negative helicity spin bundles $S^\pm$ are isomorphic to the bundles of even and odd forms $\Lambda^{\text{even/odd}} F^*$. The Clifford action on $\chi \in \Lambda^* F^*$ is given by

$$(x + \xi) \cdot \chi = i_x \chi + \xi \wedge \chi . \tag{2.6}$$

The isomorphism (2.5) is not unique but is given by a choice of volume form ϵ (though is independent of the sign of ϵ).[4] If $\chi \in \Lambda^* F^*$ we write $\chi_\epsilon \in S$ for the corresponding spinor. The usual spinor bilinear form $\psi_\epsilon^t \cdot \chi_\epsilon$ on S is then related to the Mukai pairing $\langle \cdot, \cdot \rangle$ on forms by

$$(\psi_\epsilon^t \cdot \chi_\epsilon) \epsilon = \langle \psi, \chi \rangle = \sum_p (-)^{[(p+1)/2]} \psi_p \wedge \chi_{6-p} , \tag{2.7}$$

where the subscripts denote the degree of the component forms in $\Lambda^* F^*$ and $[(p+1)/2]$ takes the integer part of $(p+1)/2$.

A metric g and B-field on F naturally define an $O(6) \times O(6)$ subgroup of $O(6,6)$ and hence a decomposition of S into $Spin(6)$-bundles $S = S_1 \otimes S_2$. The two $Spin(6)$-spinors $\eta_\pm^1$ and $\eta_\pm^2$ defined in (2.1) are naturally sections of S_1 and S_2 respectively. In terms of the diagonal $Spin(6)$ group under which we identify $S_1 \simeq S_2$, we can view $\chi_\epsilon \in S$ as a $Spin(6)$ bispinor, that is, as an element of $\text{Cliff}(6, 0; \mathbb{R})$. Explicitly one can write real $\chi_\epsilon^\pm \in S^\pm$ as

$$\chi_\epsilon^\pm = \zeta_+ \bar\zeta'_\pm \pm \zeta_- \bar\zeta'_\mp , \tag{2.8}$$

---

[4] We are using the same symbol ϵ to denote the volume form and the ten-dimensional spinors. The distinction between the two should hopefully be clear given the context.

where ζ_+, ζ'_+ are ordinary $Spin(6)$ spinors and elements of S_1^+ and S_2^+ respectively. From this perspective $\chi_\epsilon^\pm$ is a matrix. In fact it can be expanded as

$$\chi_\epsilon^\pm = \sum_p \frac{1}{8p!} \chi_{m_1\ldots m_p}^\pm \gamma^{m_1\ldots m_p}, \tag{2.9}$$

with

$$\chi_{m_1\ldots m_p}^\pm = \mathrm{tr}(\chi^\pm \gamma_{m_p\ldots m_1}) \in \Lambda^p F, \tag{2.10}$$

and where γ^m are $Spin(6)$ gamma-matrices and the trace is over the $Spin(6)$ indices. For χ_ϵ^+ only the even forms are non-zero, while for χ_ϵ^- the odd forms are non-zero. This gives an explicit realisation of the isomorphism between $S^\pm$ and $\Lambda^{\mathrm{even/odd}} F^*$ using the volume form ϵ_g compatible with the metric g.

Explicitly the $O(6,6)$ Clifford action (2.6) is realised in terms of commutators and anticommutators

$$(x + \xi) \cdot \chi_\epsilon^\pm = \tfrac{1}{2}[x^m \gamma_m, \chi_\epsilon^\pm]_\mp + \tfrac{1}{2}[\xi_m \gamma^m, \chi_\epsilon^\pm]_\pm . \tag{2.11}$$

Similarly the Mukai pairing is given by

$$\langle \psi, \chi \rangle = -8 \, \mathrm{tr}(\psi_\epsilon^t \chi_\epsilon) \epsilon_g . \tag{2.12}$$

where

$$\psi_\epsilon^t := \gamma_{(6)} C \psi_\epsilon^T C^{-1} , \tag{2.13}$$

with $\gamma^{(6)} = \frac{1}{6!} \epsilon_g^{m_1\ldots m_6} \gamma_{m_1\ldots m_6}$ and ϵ_g is the natural orientation compatible with the metric g (defined up to an arbitrary sign). (See appendix A for more details.)

Now consider the pair of complex $O(6,6)$ spinors

$$\begin{aligned}
\Phi^+ &= \mathrm{e}^{-B} \Phi_0^+ := \mathrm{e}^{-B} \eta_+^1 \bar\eta_+^2 , \\
\Phi^- &= \mathrm{e}^{-B} \Phi_0^- := \mathrm{e}^{-B} \eta_+^1 \bar\eta_-^2 ,
\end{aligned} \tag{2.14}$$

where B is the NS two-form on F and e^{-B} acts by wedge product. First one notes that when B is non-trivial, $\Phi^\pm$ are actually not quite sections of $S^\pm$. Instead one must consider the extension E

$$0 \longrightarrow F^* \longrightarrow E \longrightarrow F \longrightarrow 0, \tag{2.15}$$

defined as follows. If on the overlap of two patches $U_\alpha \cap U_\beta$ the B-field is patched by

$$B_\alpha = B_\beta + \mathrm{d}A_{\alpha\beta} \tag{2.16}$$

then in the extension (2.15) we must identify

$$x_\alpha + \xi_\alpha = x_\beta + \left(\xi_\beta + i_{x_\beta} \mathrm{d}A_{\alpha\beta}\right). \tag{2.17}$$

Since $i_{x_\alpha} \xi_\alpha = i_{x_\beta} \xi_\beta$, the $O(d,d)$ metric can still be defined on the extension E and thus one can define spinor bundles $S^\pm(E)$ and hence $\Phi^\pm \in S^\pm(E)$.

In order to introduce the notion of pure spinors we need to define the anihilator space L_Φ of an $O(6,6)$ spinor as

$$L_\Phi = \{V \in E : V \cdot \Phi = 0\} . \tag{2.18}$$

A spinor is called pure whenever its annihilator space is maximal isotropic, that is L_Φ is six-dimensional, and $\forall V, V' \in L_\Phi$, $(V, V') = 0$ holds. A pure spinor Φ therefore induces a decomposition $E = L_\Phi + \bar{L}_\Phi$. The complex $O(6,6)$ spinors $\Phi^\pm$ defined in (2.14) are pure spinors.

Individually $\Phi^\pm$ each defines an $SU(3,3)$ structure on E. Provided these structures are compatible, together they define a common $SU(3) \times SU(3)$ structure. The requirements of compatibility are that $\dim(L_{\Phi+} \cap L_{\Phi-}) = 3$, and that $\Phi^\pm$ have the same normalization [8]. In terms of Mukai pairings, they read [45]

$$\left\langle \Phi^+, V \cdot \Phi^- \right\rangle = \left\langle \bar{\Phi}^+, V \cdot \Phi^- \right\rangle = 0 \quad \forall V \in E \ , \tag{2.19}$$

$$\left\langle \Phi^+, \bar{\Phi}^+ \right\rangle = \left\langle \Phi^-, \bar{\Phi}^- \right\rangle \ . \tag{2.20}$$

If $\Phi^\pm$ are built out of $Spin(6)$ spinors in the form of eq. (2.14), they are automatically compatible [47]. The pair $\Phi^\pm$ in (2.14) therefore defines an $SU(3) \times SU(3)$ structure on E. In particular, one can see that they are invariant under independent $SU(3)$ groups acting on η^1 and η^2. Note that in terms of the local $SU(2)$ structure (2.2) they are given by [46, 47]

$$\begin{aligned}
\Phi^+ &= \frac{1}{8}\,\mathrm{e}^{-B}\left(\bar{c}_\parallel\,\mathrm{e}^{-\mathrm{i}j} - \mathrm{i}\bar{c}_\perp\,\omega\right) \wedge \mathrm{e}^{-\mathrm{i}v \wedge v'} \ , \\
\Phi^- &= -\frac{1}{8}\,\mathrm{e}^{-B}\left(c_\perp\mathrm{e}^{-\mathrm{i}j} + \mathrm{i}c_\parallel\,\omega\right) \wedge (v + \mathrm{i}v') \ ,
\end{aligned} \tag{2.21}$$

where $c_\parallel, c_\perp$ are complex functions satisfying $|c_\parallel|^2 + |c_\perp|^2 = 1$. $c_\parallel$ ($c_\perp$) vanishes when the two spinors $\eta^{1,2}$ are orthogonal (parallel), namely $\eta^2_+ = c_\parallel\,\eta^1_+ + c_\perp (v + iv')_m\gamma^m\eta^1_-$. At the points where the spinors are parallell ($c_\perp = 0$), the expression (2.21) should be understood as $\Phi^+ = \frac{1}{8}\,\mathrm{e}^{-B}\,\mathrm{e}^{-\mathrm{i}J}$, $\Phi^- = -\frac{\mathrm{i}}{8}\,\mathrm{e}^{-B}\,\Omega$, where J and Ω are the two- and three-form of the single SU(3) structure defined by the coinciding spinors. In this case, Φ^+ defines a symplectic structure, and Φ^- a complex structure. Complex and symplectic structures are particular cases of generalized complex structures. In this situation the compatibility conditions (2.19) imply the familiar requirements $J \wedge \Omega = 0, B \wedge \Omega = 0$ while the normalization condition (2.20) implies $J \wedge J \wedge J = \frac{3}{4}\mathrm{i}\Omega \wedge \bar{\Omega}$. In the general case, Φ^- contains not only a 3-form, but also a 1 and a 5-form, and defines a generalized complex structure that is not purely complex but is a mixture of complex and symplectic structures.

One key point in connecting these generalised geometrical structures to supergravity, is that, following Hitchin [2]–[4], one can show that there is a natural special Kähler structure on the space of pure spinors at a point. Furthermore, this structure precisely gives the metric for the "four-dimensional" kinetic terms in the reformulation of type II supergravity in a $N = 2$ four-dimensional-type form [45]. This structure is reviewed in the appendix B. The second key point is that the prepotentials, which describe the potential terms and gaugings of the $N = 2$ theory, are also naturally defined in terms of generalised geometrical structures. This is discussed in section 4.

Here, let us first briefly summarize the special Kähler structure. Working at a fixed point in the manifold, one starts with a real stable $Spin(6,6)$ spinor, or its associated form $\chi^\pm$. Such form is stable if it lies in an open orbit of $Spin(6,6)$. One can construct a $Spin(6,6)$ invariant six-form, known as the Hitchin function $H(\chi^\pm)$, which is homogeneous

of degree two as a function of $\chi^{\pm}$. One can get a second real form by derivation of the Hitchin function: $\hat{\chi}^{\pm}(\chi) := -\partial H(\chi^{\pm})/\partial\chi^{\pm}$. This form $\hat{\chi}^{\pm}$ has the same parity as $\chi^{\pm}$, and can be used to define the complex spinors $\Phi^{\pm} = \chi^{\pm} + \mathrm{i}\hat{\chi}^{\pm}$. Hitchin showed that the complex spinors built in this form are pure. Since H is homogeneous of degree two in $\chi^{\pm}$, we have

$$H(\Phi^{\pm}) = \frac{1}{2}\langle\chi^{\pm},\hat{\chi}^{\pm}\rangle = \mathrm{i}\langle\Phi^{\pm},\bar{\Phi}^{\pm}\rangle \ . \tag{2.22}$$

There is a symplectic structure on the space of stable spinors given by the Mukai pairing and a complex structure corresponding to the complex spinor $\Phi^{\pm}$. Both complex and symplectic structures are integrable, and therefore the space of stable forms (or pure spinors) is Kähler, or rather it is rigid special Kähler (for more details, see appendix B). Quotenting this space by the $\mathbb{C}^*$ action $\Phi^{\pm} \to \lambda\Phi^{\pm}$ for $\lambda \in \mathbb{C}^*$ (i.e., modding out by rescalings of the pure spinor), gives a space with a the Kähler potential K is related to the Hitchin function by

$$\mathrm{e}^{-K^{\pm}} = H(\Phi^{\pm}) = \mathrm{i}\langle\Phi^{\pm},\bar{\Phi}^{\pm}\rangle, \tag{2.23}$$

which defines a local special Kähler metric.

For a single $SU(3)$ structure, i.e. for $\Phi^+ = \frac{1}{8}\mathrm{e}^{-(B+\mathrm{i}J)}$, $\Phi^- = -\frac{\mathrm{i}}{8}\mathrm{e}^{-B}\Omega$, the Kähler potentials (2.23) are given respectively by the familiar expressions

$$\mathrm{e}^{-K^+} = \tfrac{1}{48}J \wedge J \wedge J, \qquad \mathrm{e}^{-K^-} = \tfrac{\mathrm{i}}{64}\Omega \wedge \bar{\Omega}. \tag{2.24}$$

Note that B drops from these expressions (which is easy to see since $\langle\mathrm{e}^{-B}\psi,\mathrm{e}^{-B}\chi\rangle = \langle\psi,\mathrm{e}^B\mathrm{e}^{-B}\chi\rangle = \langle\chi,\psi\rangle$).

In the following it will be useful to have a decomposition of $O(6,6)$ spinors under the $SU(3)\times SU(3)$ subgroup defined by Φ^+ and Φ^-. From (2.8) the decomposition of a positive chirality spinor under $Spin(6) \times Spin(6)$ is given by

$$\mathbf{32}^+ = (\mathbf{4},\mathbf{4}) + (\bar{\mathbf{4}},\bar{\mathbf{4}}) \ . \tag{2.25}$$

Under each $SU(3)$ subgroup of $Spin(6)$ we have $\mathbf{4} = \mathbf{1} + \mathbf{3}$. Hence under $SU(3) \times SU(3)$, the $O(6,6)$ spinor decomposes into 8 different representations. A similar decomposition of a negative chirality $O(6,6)$ spinor gives eight further representations. Denoting by $U_{\mathbf{r},\mathbf{s}}$ the set of forms transforming in the $(\mathbf{r},\mathbf{s})$ representation of $SU(3) \times SU(3)$ together these decompositions can be arranged in a diamond as given in table 1 [53].[5]

$U_{\mathbf{1},\bar{\mathbf{1}}}$ contains a sum of even forms while $U_{\bar{\mathbf{3}},\bar{\mathbf{1}}}$ and $U_{\mathbf{1},\mathbf{3}}$ contain a sum of odd forms. Similarly, third row consists of even forms, the forth of odd forms and so on. Note that, unlike the decomposition of forms induced by a complex structure into (p,q)-forms, the elements of $U_{\mathbf{r},\mathbf{s}}$ are not necessarily of fixed degree. Instead $U_{\mathbf{r},\mathbf{s}}$ contains forms of mixed degree which however are always even or odd. For example, for a single $SU(3)$ structure on F (which is a particular case of an $SU(3) \times SU(3)$ structure on E), a form belongs to the space $U_{\mathbf{1},\bar{\mathbf{1}}}$ if it is a multiple of $\mathrm{e}^{-(B+\mathrm{i}J)}$. Thus it indeed contains all even 0-, 2-, 4- and 6-form. Conversely forms of fixed degree are linear combinations of elements in different U's. For example, a zero-form is a linear combination of elements in $U_{\mathbf{1},\bar{\mathbf{1}}} \oplus U_{\bar{\mathbf{1}},\mathbf{1}} \oplus U_{\mathbf{3},\bar{\mathbf{3}}} \oplus U_{\bar{\mathbf{3}},\mathbf{3}}$.

---

[5]By an abuse of notation, it is convenient to use $\bar{\mathbf{1}}$ to denote the singlet coming from the decomposition of $\bar{\mathbf{4}}$.

$$
\begin{array}{ccccccc}
& & & U_{\mathbf{1},\bar{\mathbf{1}}} & & & \\
& & U_{\bar{\mathbf{3}},\bar{\mathbf{1}}} & & U_{\mathbf{1},\mathbf{3}} & & \\
& U_{\mathbf{3},\bar{\mathbf{1}}} & & U_{\bar{\mathbf{3}},\mathbf{3}} & & U_{\mathbf{1},\bar{\mathbf{3}}} & \\
U_{\bar{\mathbf{1}},\bar{\mathbf{1}}} & & U_{\mathbf{3},\mathbf{3}} & & U_{\bar{\mathbf{3}},\bar{\mathbf{3}}} & & U_{\mathbf{1},\mathbf{1}} \\
& U_{\bar{\mathbf{1}},\mathbf{3}} & & U_{\mathbf{3},\bar{\mathbf{3}}} & & U_{\bar{\mathbf{3}},\mathbf{1}} & \\
& & U_{\bar{\mathbf{1}},\bar{\mathbf{3}}} & & U_{\mathbf{3},\mathbf{1}} & & \\
& & & U_{\bar{\mathbf{1}},\mathbf{1}} & & &
\end{array}
$$

Table 1: Generalized $SU(3) \times SU(3)$ diamond.

3. Field decompositions and spectrum

In this section we discuss the group-theoretical properties of the massless type II supergravities fields in a background with a generalized tanget bundle $T^{3,1} \oplus F \oplus F^*$. In particular we show how the fields assemble in $N = 2$ — like multiplets.

If $F \oplus F^*$ admits an $SU(3) \times SU(3)$ structure all ten-dimensional fields can be decomposed under $Spin(3,1) \times SU(3) \times SU(3)$. In fact it is slightly simpler to first go to light-cone gauge and discuss the decompostion under $SO(2) \times SU(3) \times SU(3)$ instead. In order to do so let us first recall the decomposition of the two 8-dimensional inequivalent Majorana-Weyl representations $\mathbf{8_S}$ and $\mathbf{8_C}$ and the vector representation $\mathbf{8_V}$ of $SO(8)$ under $SO(8) \to SO(2) \times SO(6) \to SO(2) \times SU(3)$. One has [54]

$$
\begin{aligned}
\mathbf{8_S} &\to \mathbf{4}_{\frac{1}{2}} \oplus \overline{\mathbf{4}}_{-\frac{1}{2}} \to \mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}} \oplus \mathbf{3}_{\frac{1}{2}} \oplus \overline{\mathbf{3}}_{-\frac{1}{2}}\,, \\
\mathbf{8_C} &\to \mathbf{4}_{-\frac{1}{2}} \oplus \overline{\mathbf{4}}_{\frac{1}{2}} \to \mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}} \oplus \mathbf{3}_{-\frac{1}{2}} \oplus \overline{\mathbf{3}}_{\frac{1}{2}}\,, \\
\mathbf{8_V} &\to \mathbf{1_1} \oplus \mathbf{1_{-1}} \oplus \mathbf{6_0} \to \mathbf{1_1} \oplus \mathbf{1_{-1}} \oplus \mathbf{3_0} \oplus \overline{\mathbf{3}}_{\mathbf{0}}\,.
\end{aligned}
\tag{3.1}
$$

where the subscript denotes the helicity of $SO(2)$.

Let us start with the decomposition of the fermions which arise in the (NS,R) and (R,NS) sector. More precisly, in type IIA the two gravitinos together with the two dilatinos are in the $(\mathbf{8_S}, \mathbf{8_V})$ and $(\mathbf{8_V}, \mathbf{8_C})$ of $SO(8)_L \times SO(8)_R$ while in type IIB they come in the $(\mathbf{8_S}, \mathbf{8_V})$ and $(\mathbf{8_V}, \mathbf{8_S})$ representations. The decomposition of these representations under $SO(8)_L \times SO(8)_R \to SO(2) \times SU(3)_L \times SU(3)_R$ yields[6]

$$
\begin{aligned}
(\mathbf{8_S}, \mathbf{8_V}) \to\ & (\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\frac{3}{2},-\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-\frac{3}{2},\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{\pm\frac{1}{2}} \\
& \oplus (\mathbf{3},\mathbf{3})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{-\frac{1}{2}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{-\frac{1}{2}}\,, \\
(\mathbf{8_V}, \mathbf{8_S}) \to\ & (\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{\frac{3}{2},-\frac{1}{2}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{-\frac{3}{2},\frac{1}{2}} \\
& \oplus (\mathbf{3},\mathbf{3})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{\frac{1}{2}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{-\frac{1}{2}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{-\frac{1}{2}}\,, \\
(\mathbf{8_V}, \mathbf{8_C}) \to\ & (\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{-\frac{3}{2},\frac{1}{2}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{\frac{3}{2},-\frac{1}{2}} \\
& \oplus (\mathbf{3},\mathbf{3})_{-\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{-\frac{1}{2}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{\frac{1}{2}}\,.
\end{aligned}
\tag{3.2}
$$

---

[6]The $SO(2)$ factor in the decomposition of $SO(8)_L$ and $SO(8)_R$ is of course the same.

Exactly as we did in ref. [45] we do not consider the most general $N = 2$ — like supergravity but only keep two gravitinos in the gravitational multiplet and project out all other (possibly massive) spin-$\frac{3}{2}$ multiplets. This ensures a 'standard' N=2 theory with only the gravitational multiplet plus possibly vector, tensor and hypermultiplets. In this case the couplings in the low energy effective action are well known and highly constrained by the $N = 2$ supersymmetry.

From (3.2) we learn that keeping only the two gravitinos of the gravitational multiplet is insured if all representations of the form $(\mathbf{3},\mathbf{1})$, $(\bar{\mathbf{3}},\mathbf{1})$, $(\mathbf{1},\mathbf{3})$, $(\mathbf{1},\bar{\mathbf{3}})$ are projected out. In terms of the representations in the diamond in table 1, this amounts to keeping only the elements in the horizontal and vertical axes. This is the analogue of projecting out all triplets in the case of a single $SU(3)$ structure as we did in ref. [45]. In that case it also removed all $O(6)$ vectors (or equivalently all one-forms) from the spectrum. For a generalized $SU(3) \times SU(3)$ structure we are lead to project out the vectors of $O(6,6)$, which decompose under $SU(3) \times SU(3)$ precisely as $\mathbf{12} \to (\mathbf{3},\mathbf{1}) \oplus (\bar{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{1},\bar{\mathbf{3}})$. Note that projecting out $O(6,6)$ vectors does not imply projecting out all $O(6)$ vectors. For a generic $SU(3) \times SU(3)$ structure, there are $O(6)$ vectors (or equivalently one forms) that remain in the spectrum, as for example those contained in $U_{\mathbf{1},\mathbf{1}}$. Whenever the structure is not a single $SU(3)$, this representation, which is not projected out, contains a one-form, and the same is true for all the other representations in the horizontal axis of the diamond.

After this projection both type II theories have two gravitinos and two Weyl fermions (dilatinos) in the $(\mathbf{1},\mathbf{1})$ representations. They reside in the gravitational multiplet and the 'universal' tensor multiplet respectively. Furthermore, eq. (3.2) shows that there is a pair of Weyl fermions in the representations $(\mathbf{3},\mathbf{3}) \oplus (\bar{\mathbf{3}},\bar{\mathbf{3}})$ and a pair in the $(\bar{\mathbf{3}},\mathbf{3}) \oplus (\mathbf{3},\bar{\mathbf{3}})$. These fermions are members of vector- or hypermultiplets depending on which type II theory is being considered.

The bosonic fields in the NS sector can be similarly decomposed under $SU(3) \times SU(3)$. It is convenient to use the combination $E_{MN} = g_{MN} + B_{MN}$ of the metric and the B-field since from a string theoretical point it is a tensor product of a left and a right NS-mode excitation. As a consequence it decomposes under $SU(3) \times SU(3)$ as

$$
\begin{aligned}
E_{\mu\nu} &: (\mathbf{1},\mathbf{1})_{\pm\mathbf{2}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{T}} \,, \\
E_{\mu m} &: (\mathbf{1},\mathbf{3})_{\pm\mathbf{1}} \oplus (\mathbf{1},\bar{\mathbf{3}})_{\pm\mathbf{1}} \,, \\
E_{m\mu} &: (\mathbf{3},\mathbf{1})_{\pm\mathbf{1}} \oplus (\bar{\mathbf{3}},\mathbf{1})_{\pm\mathbf{1}} \,, \\
E_{mn} &: (\mathbf{3},\mathbf{3})_{\mathbf{0}} \oplus (\bar{\mathbf{3}},\bar{\mathbf{3}})_{\mathbf{0}} \oplus (\bar{\mathbf{3}},\mathbf{3})_{\mathbf{0}} \oplus (\mathbf{3},\bar{\mathbf{3}})_{\mathbf{0}} \,,
\end{aligned}
\tag{3.3}
$$

where $\mathbf{T}$ denotes the antisymmetric tensor. Projecting out the representations $(\mathbf{3},\mathbf{1})$, $(\bar{\mathbf{3}},\mathbf{1})$, $(\mathbf{1},\mathbf{3})$, $(\mathbf{1},\bar{\mathbf{3}})$ leaves only $E_{\mu\nu}$ and E_{mn} in the spectrum. From a four-dimensional point of view $E_{\mu\nu}$ corresponds to the graviton and an antisymmetric tensor while E_{mn} represent scalar fields. The latter can be viewed as parametrizing the deformations of the $SU(3) \times SU(3)$ structure or equivalently as deformations of the pure spinors $\Phi^{\pm}$. More precisely, keeping the normalization of the pure spinors fixed, $\delta\Phi^+$ transforms in the $(\bar{\mathbf{3}},\mathbf{3})$, while $\delta\Phi^-$ transforms in the $(\bar{\mathbf{3}},\bar{\mathbf{3}})$ (and $\delta\bar{\Phi}^+$, $\delta\bar{\Phi}^-$ transform in the complex conjugate representations, $(\mathbf{3},\bar{\mathbf{3}})$ and $(\mathbf{3},\mathbf{3})$ respectively).

multiplet	$SU(3) \times SU(3)$rep.	bosonic field content
gravity multiplet	$(\mathbf{1}, \mathbf{1})$	$g_{\mu\nu}, A_1^+$
tensor multiplet	$(\mathbf{1}, \mathbf{1})$	$B_{\mu\nu}, \phi, A_0^-$
vector multiplets	$(\mathbf{3}, \bar{\mathbf{3}})$	$A_1^+, \delta\Phi^+$
hypermultiplets	$(\mathbf{3}, \mathbf{3})$	$\delta\Phi^-, A_0^-$

Table 2: N=2 multiplets in type IIA

Finally we decompose the fields in the RR-sector. Here the bosonic fields arise from the decomposition of $(\mathbf{8}_S, \mathbf{8}_C)$ for type IIA and $(\mathbf{8}_S, \mathbf{8}_S)$ for type IIB. One finds (after projecting out the triplets)

$$
\begin{aligned}
\text{IIA}: \quad & (\mathbf{8}_S, \mathbf{8}_C) \to (\mathbf{1}, \mathbf{1})_{\pm\mathbf{1}, \mathbf{0}} \oplus (\mathbf{3}, \mathbf{3})_{\mathbf{0}} \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}})_{\mathbf{0}} \oplus (\mathbf{3}, \bar{\mathbf{3}})_{\mathbf{1}} \oplus (\bar{\mathbf{3}}, \mathbf{3})_{-\mathbf{1}}, \\
\text{IIB}: \quad & (\mathbf{8}_S, \mathbf{8}_S) \to (\mathbf{1}, \mathbf{1})_{\pm\mathbf{1}, \mathbf{0}} \oplus (\mathbf{3}, \mathbf{3})_{\mathbf{1}} \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}})_{-\mathbf{1}} \oplus (\mathbf{3}, \bar{\mathbf{3}})_{\mathbf{0}} \oplus (\bar{\mathbf{3}}, \mathbf{3})_{\mathbf{0}}.
\end{aligned}
\tag{3.4}
$$

In type IIA the RR sector contains gauge potentials of odd degree. The decomposition (3.4) naturally groups these into helicity ± 1 and helicity 0 states from a four-dimensional point of view. This leads us to define

$$
\begin{aligned}
A_0^- &= A_{(0,1)} + A_{(0,3)} + A_{(0,5)} \simeq (\mathbf{1}, \mathbf{1})_{\mathbf{0}} \oplus (\mathbf{3}, \mathbf{3})_{\mathbf{0}} \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}})_{\mathbf{0}}, \\
A_1^+ &= A_{(1,0)} + A_{(1,2)} + A_{(1,4)} + A_{(1,6)} \simeq (\mathbf{1}, \mathbf{1})_{\pm\mathbf{1}} \oplus (\mathbf{3}, \bar{\mathbf{3}})_{\mathbf{1}} \oplus (\bar{\mathbf{3}}, \mathbf{3})_{-\mathbf{1}},
\end{aligned}
\tag{3.5}
$$

where $A_{(p,q)}$ is a 'four-dimensional' p-form and a 'six-dimensional' q-form.[7] A_0^- contains 'four-dimensional' scalar degrees of freedom and is a sum of odd 'six-dimensional' forms while A_1^+ contains 'four-dimensional' vectors and is a sum even 'six-dimensional' forms.

In type IIB the situation is exactly reversed. Here we define

$$
\begin{aligned}
A_0^+ &= A_{(0,0)} + A_{(0,2)} + A_{(0,4)} + A_{(0,6)} \simeq (\mathbf{1}, \mathbf{1})_{\mathbf{0}} \oplus (\mathbf{3}, \bar{\mathbf{3}})_{\mathbf{0}} \oplus (\bar{\mathbf{3}}, \mathbf{3})_{\mathbf{0}}, \\
A_1^- &= A_{(1,1)} + A_{(1,3)} + A_{(1,5)} \simeq (\mathbf{1}, \mathbf{1})_{\mathbf{1}} \oplus (\mathbf{3}, \mathbf{3})_{\mathbf{1}} \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}})_{-\mathbf{1}}.
\end{aligned}
\tag{3.6}
$$

As expected all these fields combine into $N = 2$ multiplets, as shown in tables 2 and 3.

We see that the fields arrange nicely and (mirror) symmetrically into multiplets of a given Spin(6,6) chirality. Mirror symmetry amounts to a exchange of even and odd Spin(6,6) chirality, or to an exchange of one $\mathbf{3}$ into a $\bar{\mathbf{3}}$. This is the analogue of the exchange between $\mathbf{6} \oplus \bar{\mathbf{3}}$ and $\mathbf{8} \oplus \mathbf{1}$ proposed in [12] for a single $SU(3)$ structure. From these tables it should be clear that $SU(3) \times SU(3)$ structure is the relevant one for $N = 2$ effective actions coming from type II theories.

---

[7]There is an ambiguity in the representation of the scalar degrees of freedom arising in the RR-sector. They can be equally well written as a four-dimensional two-form. On the other hand, A_1^+ includes both the vector and dual vector degrees of freedom.

multiplet	$SU(3) \times SU(3)$rep.	bosonic field content
gravity multiplet	$(\mathbf{1}, \mathbf{1})$	$g_{\mu\nu}, A_1^-$
tensor multiplet	$(\mathbf{1}, \mathbf{1})$	$B_{\mu\nu}, \phi, A_0^+$
vector multiplets	$(\mathbf{3}, \mathbf{3})$	$A_1^-, \delta\Phi^-$
hypermultiplets	$(\mathbf{3}, \bar{\mathbf{3}})$	$\delta\Phi^+, A_0^+$

Table 3: N=2 multiplets in type IIB

4. $N = 2$ and $N = 1$ superpotentials

In this section we show that the $\mathcal{N} = 2$ Killing prepotentials and the $\mathcal{N} = 1$ superpotential found for $SU(3)$ structures in [45] have exactly the same functional form when the structure is generalized to $SU(3) \times SU(3)$.

The $N = 2$ analogue of the $N = 1$ superpotential and the $N = 1$ D-term are encoded in the Killing prepotentials $\mathcal{P}^x$, $x = 1, 2, 3$. These, together with its derivatives, determine the scalar potential [55]. The Killing prepotentials can equivalently be expressed in terms of the $SU(2)^8$ gravitino mass matrix S_{AB}, via

$$S_{AB} = \frac{\mathrm{i}}{2}\, \mathrm{e}^{\frac{1}{2}K_V} \sigma_{AB}^x \mathcal{P}^x, \qquad \sigma_{AB}^x = \begin{pmatrix} \delta^{x1} - i\delta^{x2} & -\delta^{x3} \\ -\delta^{x3} & -\delta^{x1} - i\delta^{x2} \end{pmatrix}, \tag{4.1}$$

where K_V is the Kähler potential of the vector multiplets. The gravitino mass matrix S_{AB} is obtained from the supersymmetry transformation of the four-dimensional $\mathcal{N} = 2$ gravitinos, which has the generic form

$$\delta\psi_{A\mu} = D_\mu\varepsilon_A + i\gamma_\mu S_{AB}\varepsilon^B\,, \ \ A = 1, 2 \tag{4.2}$$

The four dimensional gravitinos $\psi_{A\mu}$ are related to the ten dimensional ones Ψ_M by [45]

$$\hat{\Psi}_\mu^A := \Psi_\mu^A + \tfrac{1}{2}\Gamma_\mu{}^m\Psi_m^A = \psi_{A\mu+} \otimes \eta_\pm^A + \psi_{A\mu-} \otimes \eta_\mp^A + \ldots \tag{4.3}$$

where no sum over A is taken on the right hand side, and the $\pm$ are correlated to the chirality of the ten-dimensional spinor, that we take to be negative (positive) for $A = 1$ (2) in IIA, and negative for $A = 1, 2$ in IIB. In this expression, the dots correspond to the triplets.

---

[8]The four-dimensional $N = 2$ theory has a local $SU(2)_R$ symmetry which rotates the two (complex) gravitinos $\psi_{A\mu}$ into each other. In ten dimensions it arises from the $O(2)$ rotation of the two ten-dimensional Majorana-Weyl fermions into each other.

The supersymmetry transformation of the gravitinos for the democratic formulation [56] in Einstein frame is

$$\delta\Psi_M = D_M\epsilon - \frac{1}{96}\mathrm{e}^{-\phi/2}\left(\Gamma_M{}^{PQR}H_{PQR} - 9\Gamma^{PQ}H_{MPQ}\right)P\epsilon$$
$$- \sum_n \frac{\mathrm{e}^{(5-n)\phi/4}}{64\,n!}\left[(n-1)\Gamma_M{}^{N_1\ldots N_n} - n(9-n)\delta_M{}^{N_1}\Gamma^{N_2\ldots N_n}\right]F_{N_1\ldots N_n}\,P_n\,\epsilon. \quad (4.4)$$

In this expression, $n = 0, 2, 4, 6, 8$, $P = \Gamma_{11}$ and $P_n = (\Gamma_{11})^{n/2}\sigma^1$ for IIA. For IIB we have instead a sum over $n = 1, 3, 5, 7, 9$, $P = -\sigma^3$ and $P_n = i\sigma^2$ for $n = 1, 5, 9$ and $P_n = \sigma^1$ for $n = 3, 7$.

In order to get S_{AB}, we need to project the supersymmetry transformation of the ten-dimensional shifted gravitino $\delta\hat{\Psi}_\mu$ onto the $SU(3)$-singlet parts. The relevant projector for type IIB is

$$\Pi = \begin{pmatrix} \Pi^1_- \\ \Pi^2_- \end{pmatrix} = \begin{pmatrix} \mathbf{1}\otimes\eta^1_-\bar{\eta}^1_- \\ \mathbf{1}\otimes\eta^2_-\bar{\eta}^2_- \end{pmatrix} \quad (4.5)$$

(we are using $\bar{\eta}^A_\pm\eta^A_\pm = 1$). For type IIA, we have instead Π^1_- and $\Pi^2_+ = \mathbf{1}\otimes\left(\eta^2_+\otimes\bar{\eta}^2_+\right)$. In the following we show the details of the type IIB calculation but only give the results for type IIA since it follows straightforwardly.

Inserting the projector (4.5) in $\delta\hat{\Psi}_\mu$, we get

$$\begin{pmatrix} \delta\psi^1_{\mu+} \\ \delta\psi^2_{\mu+} \end{pmatrix} = \begin{pmatrix} D_\mu\epsilon^1_+ \\ D_\mu\epsilon^2_+ \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \gamma_\mu\epsilon^1_-\ \bar{\eta}^1_-\gamma^m D_m\eta^1_+ \\ \gamma_\mu\epsilon^2_-\ \bar{\eta}^2_-\gamma^m D_m\eta^2_+ \end{pmatrix} + \frac{1}{48}\begin{pmatrix} \gamma_\mu\epsilon^1_-\ H_{pqr}\ \bar{\eta}^1_-\gamma^{pqr}\eta^1_+ \\ -\gamma_\mu\epsilon^2_-\ H_{pqr}\ \bar{\eta}^2_-\gamma^{pqr}\eta^2_+ \end{pmatrix}$$
$$-\frac{1}{8}\begin{pmatrix} -\gamma_\mu\epsilon^2_-\ e^\phi\frac{1}{n!}F^-_{i_1\ldots i_n}\ \bar{\eta}^1_-\gamma^{i_1\ldots i_n}\eta^2_+ \\ \gamma_\mu\epsilon^1_-\ e^\phi\frac{1}{n!}\sigma(F^-)_{i_1\ldots i_n}\ \bar{\eta}^2_-\gamma^{i_1\ldots i_n}\eta^1_+ \end{pmatrix}, \quad (4.6)$$

where we have written the expressions in terms of string frame metric $g = e^{\phi/2}g_E$. Furthermore $F^- = F_1 + F_3 + F_5$ is the sum of odd internal RR field strengths, and $\sigma(F^-) = -F_1 + F_3 - F_5$ is the combination of forms that appears in the Mukai pairing, eq. (2.7) ($\sigma(F_\epsilon) = F^T_\epsilon$ in the spinor language) . From this we read off

$$S_{11} = \frac{\mathrm{i}}{2}\bar{\eta}^1_-\gamma^m D_m\eta^1_+ - \frac{\mathrm{i}}{48}H_{pqr}\bar{\eta}^1_-\gamma^{pqr}\eta^1_+,$$
$$S_{22} = \frac{\mathrm{i}}{2}\bar{\eta}^2_-\gamma^m D_m\eta^2_+ + \frac{\mathrm{i}}{48}H_{pqr}\bar{\eta}^2_-\gamma^{pqr}\eta^2_+,$$
$$S_{12} = \frac{\mathrm{i}}{8}e^\phi\frac{1}{n!}F^-_{i_1\ldots i_n}\bar{\eta}^1_-\gamma^{i_1\ldots i_n}\eta^2_+,$$
$$S_{21} = \frac{\mathrm{i}}{8}e^\phi\frac{1}{n!}\sigma(F)^-_{i_1\ldots i_n}\bar{\eta}^2_-\gamma^{i_1\ldots i_n}\eta^1_+. \quad (4.7)$$

Multiplying by a volume form ϵ and using (2.12), we can write these expressions in terms of Mukai pairings. S_{12} is the simplest one,

$$S_{12}\,\epsilon = S_{21}\,\epsilon = \mathrm{i}\,\mathrm{tr}\,(\eta^2_+\bar{\eta}^1_-\gamma^{i_1\ldots i_n})\frac{1}{n!}F^-_{i_1\ldots i_n}\epsilon = -\mathrm{tr}((\Phi^-_0)^t_\epsilon F^-_\epsilon)\,\epsilon$$
$$= \frac{1}{8}\langle\Phi^-_0, F^-\rangle = \frac{1}{8}\langle\Phi^-, G^-\rangle, \quad (4.8)$$

where Φ_0^- is defined in (2.14). In the third equality we have used $(\Phi_0^-)_\epsilon^t = i\eta_+^2 \bar{\eta}_-^1$ (see appendix A for more details) and we recall that F is related to F_ϵ by (2.9). In the first equality, we use $\sigma(F)_\epsilon = F_\epsilon^T$. Finally, in the last equality we have defined the RR flux G through

$$F = \mathrm{d}C - H \wedge C = \mathrm{e}^B G . \tag{4.9}$$

G is the flux for the potentials A used in the previous section, namely

$$G^+ = \mathrm{d}A_0^-, \qquad G^- = \mathrm{d}A_0^+. \tag{4.10}$$

which implies that A is related to C by $C = \mathrm{e}^B A$.

The diagonal pieces in S_{AB} require a bit more work. It is easier to show that they can also be expressed in terms of Mukai pairings by working backwards, i.e. starting from the latter and arriving at the bilinears in (4.7). Using the relation (2.11) we have

$$
\begin{aligned}
\mathrm{d}\Phi_0^+ &= \tfrac{1}{2}\left[\gamma^m, D_m(\eta_+^1 \bar{\eta}_+^2)\right]_+ \\
&= \tfrac{1}{2}\left[(\gamma^m D_m \eta_+^1)\bar{\eta}_+^2 + (\gamma^m \eta_+^1)(D_m \bar{\eta}_+^2) + (D_m \eta_+^1)(\bar{\eta}_+^2 \gamma^m) + \eta_+^1 (D_m \bar{\eta}_+^2 \gamma^m)\right] .
\end{aligned}
\tag{4.11}
$$

Similarly

$$H \wedge \Phi_0^+ = \tfrac{1}{48} H_{mnp}\left[\gamma^{mnp}\eta_+^1 \bar{\eta}_+^2 + 3\gamma^{mn}\eta_+^1 \bar{\eta}_+^2 \gamma^p + 3\gamma^m \eta_+^1 \bar{\eta}_+^2 \gamma^{np} + \eta_+^1 \bar{\eta}_+^2 \gamma^{mnp}\right]. \tag{4.12}$$

Now we have by chirality and the symmetry of the gamma matrices

$$\bar{\eta}_- \eta_+ = \bar{\eta}_- \gamma^m \eta_+ = \bar{\eta}_+ \gamma^m \eta_+ = 0. \tag{4.13}$$

Hence, we have

$$
\begin{aligned}
\tfrac{1}{8}\left\langle \Phi^-, \mathrm{d}\Phi^+\right\rangle &= \tfrac{1}{8}\left\langle \Phi_0^-, (\mathrm{d}\Phi_0^+ - H \wedge \Phi_0^+)\right\rangle = -\operatorname{tr}\left[\left(\Phi_0^-\right)_\epsilon^t \left(\mathrm{d}\Phi_0^+ - H \wedge \Phi_0^+\right)_\epsilon\right] \epsilon \\
&= -\left[\tfrac{i}{2}\bar{\eta}_-^1 \gamma^m D_m \eta_+^1 - \tfrac{i}{48} H_{mnp}\bar{\eta}_-^1 \gamma^{mnp}\eta_+^1\right]\epsilon = -S_{11}\,\epsilon,
\end{aligned}
\tag{4.14}
$$

where only the first terms in (4.11) and (4.12) survive. Similarly, one shows that

$$\tfrac{1}{8}\left\langle \Phi^-, \mathrm{d}\bar{\Phi}^+\right\rangle = S_{22}\,\epsilon \tag{4.15}$$

where now the last terms of the expressions (4.11) and (4.12) corresponding to $\bar{\Phi}_0^+$ are the only ones that survive when inserted in the Mukai pairing.

Collecting all the pieces together, we get for the matrix S_{AB} in type IIB

$$S_{AB}^{(4)}(\mathrm{IIB}) = \frac{1}{8}\mathrm{e}^{\frac{1}{2}K^-}\begin{pmatrix} -\mathrm{e}^{\frac{1}{2}K^+ + \phi^{(4)}}\left\langle \Phi^-, \mathrm{d}\Phi^+\right\rangle & -\mathrm{e}^{2\phi^{(4)}}\left\langle \Phi^-, G^-\right\rangle \\ -\mathrm{e}^{2\phi^{(4)}}\left\langle \Phi^-, G^-\right\rangle & \mathrm{e}^{\frac{1}{2}K^+ + \phi^{(4)}}\left\langle \Phi^-, \mathrm{d}\bar{\Phi}^+\right\rangle \end{pmatrix} . \tag{4.16}$$

In this expression the superscript (4) indicates that in (4.2) we are using the natural metric on $T^{1,3}$: $g_{\mu\nu}^{(4)} = e^{-2\phi^{(4)}}g_{\mu\nu}$. The four dimensional dilaton $\phi^{(4)}$ is related to the ten dimensional one and the string frame metric by $\phi^{(4)} = \phi - \tfrac{1}{4}\ln\det g_{mn}$. The Kähler

potentials $K^\pm$ are defined in (2.23) and we have used that all the six-forms are related by the normalization condition

$$\epsilon = \mathrm{i}\langle\Phi^\pm, \bar{\Phi}^\pm\rangle = e^{-K^\pm} = e^{-2\phi^{(4)}+2\phi} \ . \tag{4.17}$$

Note that S_{AB} is naturally a section of $(\Lambda^6 F^*)^{-1/2}$.

The calculation for type IIA follows straightforwardly, and gives

$$S_{AB}^{(4)}(\text{IIA}) = \frac{1}{8}\, e^{\frac{1}{2}K^+} \begin{pmatrix} e^{\frac{1}{2}K^-+\phi^{(4)}}\left\langle\Phi^+,\mathrm{d}\Phi^-\right\rangle & e^{2\phi^{(4)}}\left\langle\Phi^+,G^+\right\rangle \\[2mm] e^{2\phi^{(4)}}\left\langle\Phi^+,G^+\right\rangle & -e^{\frac{1}{2}K^-+\phi^{(4)}}\left\langle\Phi^+,\mathrm{d}\bar{\Phi}^-\right\rangle \end{pmatrix} . \tag{4.18}$$

The gravitino mass matrices obtained have excately the same functional form in terms of $\Phi^\pm$ as the one obtained in [45] for a single $SU(3)$ structure, confirming the claim made there.[9] They are symmetric under the mirror exchange $\Phi^+ \leftrightarrow \Phi^-$, $G^+ \leftrightarrow G^-$.

Given the $\mathcal{N} = 2$ Killing prepotentials, the computation of the $\mathcal{N} = 1$ superpotential is exactly the same as for a single $SU(3)$ structure. We will therefore not show the details, worked out in [45], but just quote the result[10]

$$\mathcal{W}_{\text{IIA}} = \cos^2\alpha\, e^{\mathrm{i}\beta}\langle\Phi^+,\mathrm{d}\Phi^-\rangle \,-\, \sin^2\alpha\, e^{-\mathrm{i}\beta}\langle\Phi^+,\mathrm{d}\bar{\Phi}^-\rangle \,+\, \sin 2\alpha\, e^\phi\left\langle\Phi^+,G^+\right\rangle, \tag{4.19}$$

and

$$\mathcal{W}_{\text{IIB}} = -\cos^2\alpha\, e^{\mathrm{i}\beta}\left\langle\Phi^-,\mathrm{d}\Phi^+\right\rangle \,+\, \sin^2\alpha\, e^{-\mathrm{i}\beta}\left\langle\Phi^-,\mathrm{d}\bar{\Phi}^+\right\rangle \,-\, \sin 2\alpha\, e^\phi\left\langle\Phi^-,G^-\right\rangle. \tag{4.20}$$

where α and β parameterize the $U(1)_R \in SU(2)_R$ of $\mathcal{N} = 1$, namely the $\mathcal{N} = 1$ supersymmetry parameter ε is given in terms of the $\mathcal{N} = 2$ parameters ε_A by

$$\varepsilon_A = \varepsilon n_A, \qquad n_A = \begin{pmatrix} a \\ b \end{pmatrix}, \quad a = \cos\alpha\, e^{-\frac{i}{2}\beta}, \quad b = \sin\alpha\, e^{\frac{i}{2}\beta} \ . \tag{4.21}$$

(We have absorbed the factors of $1/8$ in (4.16), (4.18) in the definition of $\mathcal{W}$.) The difference between the $SU(3) \times SU(3)$ and $SU(3)$ superpotential is in the form of the pure spinors, which leads to the appearance of new terms involving the five-form $\mathrm{d}\Phi_4^+$. As we will see in the next section, these are the mirrors of magnetic fluxes missing in pure $SU(3)$ structure constructions.

5. Mirror of magnetic fluxes

Thus far we rewrote the ten-dimensional type II supergravity in a background which admits an $SU(3) \times SU(3)$ structure. In this section we consider an actual compactification so that

---

[9]The differences in factors are due to different conventions for the normalizations of the spinors, while S_{11} and S_{22} in type IIA are interchanged with respect to the expressions in [45] because we have taken opposte conventions for the chiralities of the type IIA spinors.

[10]For orientifold compactification on $SU(3) \times SU(3)$ manifolds the superpotential has been computed in [48] by reducing the ten-dimensional gravitino mass term.

the background $M^{9,1} = M^{3,1} \times M^6$ where M^6 is a compact manifold with $SU(3) \times SU(3)$ structure. Such reductions in the special case of a pure $SU(3)$ structure were discussed in ref. [45]. The analysis here is completely analogous and therefore we only briefly review this step. In addition, we will truncate the degrees of freedom in the forms $\Phi^\pm$ to a finite dimensional space, giving a conventional effective $N = 2$ supergravity theory on $M^{3,1}$. In the case of the Calabi-Yau this truncation translates into keeping only harmonic forms and describes the moduli of the Calabi-Yau manifold. As we will see, in general situations, it is more complicated. This is discussed in section 5.2 as well as the appendix B.

The generic case will be considered in the next section. In this section we concentrate on a particular subclass of compactifications for which one obtains the mirror dual of compactifications on Calabi-Yau manifolds with magnetic H_3-flux. This case was missing in refs. [11, 45] and as a consequence the final results were not mirror symmetric. Here we close this gap and suggest a completely mirror symmetric background. Related work has been performed in refs. [23, 25, 48] and we comment on the relation in section 7.

By way of comparison we first briefly consider the case of compactification on a Calabi-Yau manifold with generic H_3-flux in the language of generalised structures and identify the truncation. We then discuss the analogous structure for the mirror symmetric background.

5.1 Generalised geometry and H_3-flux

Let us review the derivation of the low-energy effective action arising from a compactification on a Calabi-Yau manifold M_6 with general H_3-flux [57–59, 19, 60, 61, 22, 45].

One starts by identifying the moduli. Since we want to consider non-trivial H_3 flux we first split the (local) potential B into flux and moduli pieces

$$B = B^{\text{fl}} + \tilde{B}, \qquad \mathrm{d}B^{\text{fl}} = H_3, \quad \mathrm{d}\tilde{B} = 0. \tag{5.1}$$

The usual Calabi-Yau moduli correspond to expanding the Kähler form J, the modulus part $\tilde{B}$ and the holomorphic three-form Ω on M_6 in terms of forms which are harmonic with respect to the metric defined by the $SU(3)$ structure (J, Ω).

Specially one expands the three-form Ω in terms of a symplectic basis of harmonic three-forms

$$\alpha_I^{(0)}, \beta^{(0)I} \in H^3(M_6, \mathbb{R}), \qquad I = 0, \ldots, h_{2,1}, \tag{5.2}$$

with

$$\int_{M_6} \left\langle \alpha_I^{(0)}, \beta^{(0)J} \right\rangle = \delta_I{}^J, \tag{5.3}$$

and all other pairings vanishing, where we have written the symplectic structure in terms of the Mukai pairing $\langle \cdot, \cdot \rangle$. One similarly introduces a set of even harmonic forms to expand J and $\tilde{B}$:

$$
\begin{aligned}
\omega_0^{(0)} &= 1 \in H^0(M_6, \mathbb{R}), & \omega_a^{(0)} &\in H^2(M_6, \mathbb{R}), \\
\tilde{\omega}^{(0)0} &\in H^6(M_6, \mathbb{R}), & \tilde{\omega}^{(0)a} &\in H^4(M_6, \mathbb{R}),
\end{aligned}
\tag{5.4}
$$

with $a = 1, \ldots, h_{1,1}$ and

$$\int_{M_6} \left\langle \omega_A^{(0)}, \tilde{\omega}^{(0)B} \right\rangle = \delta_A{}^B, \qquad A, B = 0, \ldots, h_{1,1}, \tag{5.5}$$

and all other pairings vanishing. Explicitly, the complex Kähler form is expanded as $\tilde{B} + \mathrm{i}J = t^a\omega_a^{(0)}$. Note that the condition $J \wedge \Omega = 0$ implies that

$$\omega_a^{(0)} \wedge \alpha_A^{(0)} = \omega_a^{(0)} \wedge \beta^{(0)A} = 0 \qquad \forall a, A. \tag{5.6}$$

which is satisfied identically for harmonic forms.

It is a standard result that there are natural local special Kähler metrics on the moduli spaces of $B + \mathrm{i}J$ and Ω. These describe the kinetic energy terms of the moduli in the effective four-dimensional $N = 2$ theory. The properties of special Kähler metrics are discussed in appendix B. In general they are determined by a holomorphic prepotential F. In the Calabi-Yau context, for the Kähler moduli, introducing homogeneous complex coordinates $X^0 = c$ and $X^a = -ct^a$ the corresponding pure spinor can be written as

$$\mathrm{e}^{-\tilde{B}}\Phi_0^+ = c\mathrm{e}^{-\tilde{B}-\mathrm{i}J} = X^A\omega_A^{(0)} - F_A\tilde{\omega}^{(0)A}, \tag{5.7}$$

where $F_A = \partial F/\partial X^A$. Similarly, one has homogeneous complex coordinates Z^I for the complex structure moduli such that the pure spinor corresponding to Ω has the form

$$\mathrm{e}^{-\tilde{B}}\Phi_0^- = -\tfrac{\mathrm{i}}{8}\,\Omega = Z^I\alpha_I^{(0)} - \mathcal{F}_I\beta^{(0)I}, \tag{5.8}$$

where again $\mathcal{F}_I = \partial\mathcal{F}/\partial Z^I$. Using (5.6) one notes that $\mathrm{e}^{-\tilde{B}}\Phi_0^- = \Phi_0^-$. The corresponding Kähler potentials are given by

$$\begin{aligned}
\mathrm{e}^{-K^+} &= \mathrm{i}\int_{M^6}\left\langle\Phi_0^+, \bar{\Phi}_0^+\right\rangle = \tfrac{c^2}{48}\int_{M^6} J \wedge J \wedge J = \mathrm{i}\left(\bar{X}^A F_A - X^A \bar{F}_A\right), \\
\mathrm{e}^{-K^-} &= \mathrm{i}\int_{M^6}\left\langle\Phi_0^-, \bar{\Phi}_0^-\right\rangle = \tfrac{\mathrm{i}}{64}\int_{M^6}\Omega \wedge \bar{\Omega} = \mathrm{i}\left(\bar{Z}^I\mathcal{F}_I - Z^I\bar{\mathcal{F}}_I\right).
\end{aligned} \tag{5.9}$$

In deriving the low-energy effective action we assume that the flux H_3 also satisfied the Bianchi identity and equations of motion, and hence is also harmonic. This means

$$H_3 = \mathrm{d}B^{\mathrm{fl}} = -m^I\alpha_I^{(0)} + e_I\beta^{(0)I} \tag{5.10}$$

where m^I are the "magnetic" fluxes and e_I the "electric" fluxes. Note that for a consistent string theory background the charges m^I and e_I must be integral.

Now in the general expressions for the superpotentials given section 4 the pure spinors $\Phi^\pm$ were twisted by the full potential $B = B^{\mathrm{fl}} + \tilde{B}$. It is then natural to introduce a twisted basis of forms. We write

$$\begin{aligned}
\Phi^+ &= \mathrm{e}^{-B}\Phi_0^+ = X^A\omega_A - F_A\tilde{\omega}^A, \\
\Phi^- &= \mathrm{e}^{-B}\Phi_0^- = Z^I\alpha_I - \mathcal{F}_I\beta^I,
\end{aligned} \tag{5.11}$$

where the twisted basis forms are given by

$$\begin{aligned}
\omega_A &= \mathrm{e}^{-B^{\mathrm{fl}}}\omega_A^{(0)}, & \tilde{\omega}^A &= \mathrm{e}^{-B^{\mathrm{fl}}}\tilde{\omega}^{(0)A}, \\
\alpha_I &= \mathrm{e}^{-B^{\mathrm{fl}}}\alpha_I^{(0)}, & \beta^I &= \mathrm{e}^{-B^{\mathrm{fl}}}\beta^{(0)I}.
\end{aligned} \tag{5.12}$$

Note that $(\omega_A, \tilde{\omega}^A)$ and (α_I, β^I) are no longer of pure degree. Since the Mukai pairing is invariant under $O(6,6)$ transformations we still have the symplectic structure

$$\int_{M_6} \langle \omega_A, \tilde{\omega}^B \rangle = \delta_A{}^B, \qquad \int_{M_6} \langle \alpha_I, \beta^J \rangle = \delta_I{}^J, \tag{5.13}$$

with the other pairings vanishing. The Kähler potentials $K^\pm = -\ln i \int_{M^6} \langle \Phi^\pm, \bar{\Phi}^\pm \rangle$ are similarly still given by (5.9). Note that this twisted basis is an example of a generic truncation, satisfying the necessary conditions discussed in appendix B.

Crucially the new basis forms are no longer closed. Using the conditions (5.6), we find that the only non-zero terms are

$$\begin{aligned}
\mathrm{d}\omega_0 &= -\mathrm{e}^{-B^{\mathrm{fl}}} H_3 \wedge \omega_0^{(0)} = \mathrm{e}^{-B^{\mathrm{fl}}} (m^I \alpha_I^{(0)} - e_I \beta^{(0)I}), \\
\mathrm{d}\alpha_I &= -\mathrm{e}^{-B^{\mathrm{fl}}} H_3 \wedge \alpha_I^{(0)} = \mathrm{e}^{-B^{\mathrm{fl}}} (m^J \alpha_J^{(0)} - e_J \beta^{(0)J}) \wedge \alpha_I^{(0)}, \\
\mathrm{d}\beta^I &= -\mathrm{e}^{-B^{\mathrm{fl}}} H_3 \wedge \beta^{(0)I} = \mathrm{e}^{-B^{\mathrm{fl}}} (m^J \alpha_J^{(0)} - e_J \beta^{(0)J}) \wedge \beta^{(0)I}.
\end{aligned} \tag{5.14}$$

Let us introduce a notation "$\sim$" to denote equality up to terms which vanish under the symplectic pairing (5.13) with any basis form. The non-zero terms are then given by

$$\mathrm{d}\omega_0 \sim m^I \alpha_I - e_I \beta^I, \qquad \mathrm{d}\alpha_I \sim e_I \tilde{\omega}^0, \qquad \mathrm{d}\beta^I \sim m^I \tilde{\omega}^0, \tag{5.15}$$

where we have used (5.5), and where the first expression is actually an equality.

The corresponding low-energy effective action of Calabi-Yau compactifications with electric and magnetic fluxes has been derived in refs. [57–59, 19, 60, 61, 22, 45] and for later reference we recall the Killing prepotentials computed in [45] here. For type IIA one has

$$\begin{aligned}
\mathcal{P}^1 + \mathrm{i}\mathcal{P}^2 &= \tfrac{\mathrm{i}}{4} \mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}} \int_{M^6} \Omega \wedge H_3 = -2\mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}} \left(e_I Z^I - \mathcal{F}_I m^I \right), \\
\mathcal{P}^3 &= \mathrm{e}^{2\phi^{(4)}} \int_{M^6} A_3 \wedge H_3 = \mathrm{e}^{2\phi^{(4)}} \left(\xi^I e_I + \tilde{\xi}_I m^I \right).
\end{aligned} \tag{5.16}$$

In type IIB one finds instead

$$\begin{aligned}
\mathcal{P}^1 - \mathrm{i}\mathcal{P}^2 &= -\tfrac{\mathrm{i}}{4} \mathrm{e}^{\frac{1}{2}K_+ + \phi^{(4)}} \int_{M^6} \Omega \wedge H_3 = -2\mathrm{e}^{\frac{1}{2}K_+ + \phi^{(4)}} \left(e_I Z^I - \mathcal{F}_I m^I \right), \\
\mathcal{P}^3 &= -\tfrac{\mathrm{i}}{8} \mathrm{e}^{2\phi^{(4)}} \xi^0 \int_{M^6} \Omega \wedge H_3 = \mathrm{e}^{2\phi^{(4)}} \xi^0 \left(e_I Z^I - \mathcal{F}_I m^I \right),
\end{aligned} \tag{5.17}$$

where ξ^0 is the RR scalar of type IIB.

To summarize, we have reformulated the moduli and flux expansion in the conventional Calabi-Yau compactification in terms of a slightly modified set of twisted forms which naturally include the H_3-flux and are appropriate to the generalised geometry. A key point is that the elements of the new bases are neither of pure degree nor are closed. As we will see in the next section, this provides a very natural ansatz for the corresponding expansion for the mirror geometries.

5.2 Generalised geometry and the mirror of H_3-flux

Following the setup of ref. [45] and in analogy with our reformulation of the Calabi-Yau compactification with H_3-flux, we now look for some basis of forms on M^6 in which to expand the fields of the ten-dimensional background (summarized in tables 2 and 3). It is clear from the Calabi-Yau discussion that in general the basis forms in $\Lambda^* T M^*$ need not be of pure degree, nor closed.

Physically we are keeping only certain modes in the entire tower of Kaluza-Klein excitations which correspond to the light modes of the compactification. Obviously to actually identify this hierarchy of excitations requires a knowledge of the particular properties of M_6. In the following, rather than fix the manifold and show that there is a sensible set of light modes, we will simply assume there is such an expansion and discuss its consistency conditions. (For a further discussion of when such a truncation exists see [50].) Indeed, if mirror symmetry can be defined for a Calabi-Yau compactification with H_3 flux, then there must be some dual compactification for which such a hierarchical expansion can be identified.

The general truncation consistency conditions are discussed in detail in appendix B.[11] Since $\Phi^\pm$ and $G^\pm$ are sums of either odd or even forms, our basis should similarly be in terms of odd or even forms. For the kinetic terms to make sense (and to have the correct multiplet structure) we better ensure that the special Kähler geometry for the untruncated $\Phi^\pm$ descends to a special Kähler geometry for the finite number of modes we are keeping.

In general we identify two finite-dimensional subspaces $U^\pm \subset C^\infty(S^\pm(E))$ and require $\Phi^\pm$ to lie in $U^\pm$. Explicitly we can expand $\Phi^\pm$ in terms of a basis of forms

$$
\begin{aligned}
\Sigma^+ &= \{\omega_A, \tilde\omega^B\}, \quad A = 0, \ldots, b^+, \\
\Sigma^- &= \{\alpha_I, \beta^J\}, \quad\ \ I = 0, \ldots, b^-.
\end{aligned}
\tag{5.18}
$$

which define a symplectic structure

$$
\int_{M^6} \langle \omega_A, \tilde\omega^B \rangle = \delta_A{}^B\,, \qquad \int_{M^6} \langle \alpha_I, \beta^J \rangle = \delta_I{}^J\,,
\tag{5.19}
$$

with all other pairings vanishing. For there to be a natural local special Kähler structure on $U^\pm/\mathbb{C}^*$, these bases must satisfy a number of other conditions given in detail in appendix B. Ignoring the compatibility condition (2.19) one can then introduce holomorphic coordinates and prepotentials as before, and expand the pure spinors $\Phi^\pm$ as follows

$$
\Phi^+ = X^A \omega_A - F_A \tilde\omega^A\,, \qquad \Phi^- = Z^I \alpha_I - \mathcal{F}_I \beta^I\,.
\tag{5.20}
$$

Generically, however, the compatibility condition (2.19) imposes a relation between the moduli. To avoid this, we will assume, that (2.19) is satisfied by each pair of basis forms

$$
\langle \omega_A, V \cdot \alpha_I \rangle = \langle \omega_A, V \cdot \beta^I \rangle = \langle \tilde\omega^A, V \cdot \alpha_I \rangle = \langle \tilde\omega^A, V \cdot \beta^I \rangle = 0\,,
\tag{5.21}
$$

for all $V = x + \xi \in E$. These are the analogues of the conditions (5.6) in the Calabi-Yau case and imply that the expressions (5.20) are valid without constraining the moduli. In

---

[11]The conditions in the special case of a generic $SU(3)$ structure were also analysed recently in [50].

fact (5.21) further implies that there are no triplet representations under $SU(3) \times SU(3)$ in the expansion which has to hold so that no additional spin-$\frac{3}{2}$ multiplets are in the light spectrum. To see this, note, first, that a generic $\chi \in S(E)$ contains eight triplet components as indicated in table 1. Similarly, a generic vector $V \in E$ decomposes into four triplets $(\mathbf{3},\mathbf{1}) + (\bar{\mathbf{3}},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{1},\bar{\mathbf{3}})$ under $SU(3) \times SU(3)$. Since the Mukai pairing and the pure spinors $\Phi^{\pm}$ are singlets, the condition

$$\left\langle \Phi^+, V \cdot \chi \right\rangle = \left\langle \Phi^-, V \cdot \chi \right\rangle = 0, \quad \forall V \in E, \tag{5.22}$$

is equivalent to setting the eight triple components of χ to zero. Given the expansion (5.18) and using the fact that $\langle \psi, V \cdot \chi \rangle = -\langle \chi, V \cdot \psi \rangle$, it is easy to check that (5.22) is indeed satisfied for every basis form.

The truncated Kähler potentials are given by the same expressions as in the Calabi-Yau case (5.9) and read

$$
\begin{aligned}
\mathrm{e}^{-K^+} &= \mathrm{i} \int_{M^6} \left\langle \Phi^+, \bar{\Phi}^+ \right\rangle = \mathrm{i} \left(\bar{X}^A F_A - X^A \bar{F}_A \right) , \\
\mathrm{e}^{-K^-} &= \mathrm{i} \int_{M^6} \left\langle \Phi^-, \bar{\Phi}^- \right\rangle = \mathrm{i} \left(\bar{Z}^I \mathcal{F}_I - Z^I \bar{\mathcal{F}}_I \right) .
\end{aligned}
\tag{5.23}
$$

For the Ramond-Ramond fields we expand the combinations $A_0^{\pm}$ and $A_1^{\pm}$ defined in eqs. (3.6), (3.5) in terms of the symplectic basis (5.19) as follows

$$
\begin{aligned}
A_0^+ &= \xi^A \omega_A + \tilde{\xi}_B \tilde{\omega}^B , & A_1^- &= A_1^I \alpha_I + \tilde{A}_{1J} \beta^J , \\
A_0^- &= \xi^I \alpha_I + \tilde{\xi}_J \beta^J , & A_1^+ &= A_1^A \omega_A + \tilde{A}_{1B} \tilde{\omega}^B .
\end{aligned}
\tag{5.24}
$$

ξ^A and $\tilde{\xi}_B$ are scalars and A_1^I and $\tilde{A}_{1J}$ are vectors in type IIB while ξ^I, $\tilde{\xi}_J$, A_1^A and $\tilde{A}_{1B}$ are scalars and vectors of type IIA respectively. In the following it will sometimes be more convenient to dualize the scalars of A_0^+ and A_0^- to antisymmetric tensors and, when appropriate, discuss the effective theory in terms of them.[12] Thus we define

$$A_2^+ = \tilde{C}_2^A \omega_A + C_{2B} \tilde{\omega}^B , \qquad A_2^- = \tilde{C}_2^I \alpha_I + C_{2J} \beta^J , \tag{5.25}$$

where from a four-dimensional point of view A_2^+ is dual to A_0^+ and A_2^- is dual to A_0^-. At the level of the four-dimensional fields the duality relates

$$\xi^A \leftrightarrow C_{2A} , \qquad \tilde{\xi}_B \leftrightarrow \tilde{C}_2^B , \qquad \xi^I \leftrightarrow C_{2I} , \qquad \tilde{\xi}_J \leftrightarrow \tilde{C}_2^J . \tag{5.26}$$

The goal of this section is to find the dual of the magnetic fluxes. We know that mirror symmetry essentially exchanges $\Lambda^{\mathrm{even}} T^* M_6$ and $\Lambda^{\mathrm{odd}} T^* M_6$. We also showed in the previous section that the H_3 flux is naturally incorporated in the generalised geometry picture as non-closed basis forms (5.14). Thus for the mirror compactification it is natural

---

[12]The reason is that the magnetic fluxes or torsion charges generate masses for some of the antisymmetric tensors and the discussion becomes a bit more involved in terms of scalar degrees of freedom [19, 61, 62].

to take the same differential conditions (5.14) but with the roles of odd and even forms reversed:[13]

$$\mathrm{d}\alpha_0 \sim p^A \omega_A + e_A \tilde{\omega}^A \,, \qquad \mathrm{d}\omega_A \sim -e_A \beta^0 \,, \qquad \mathrm{d}\tilde{\omega}^A \sim p^A \beta^0 \,. \qquad (5.27)$$

Note that as before these relations are only up to terms which vanish under the symplectic pairing (5.19). Here we have singled out two of the basis forms α_0 and β^0. This is a familiar property of local special Kähler metrics. The point is that the $\Phi^\pm$ are only defined up to complex rescalings. From eqs. (5.23) we see that $\Phi^\pm \to c^\pm \Phi^\pm$ amounts to a Kähler transformation of $K^\pm$. Therefore it is possible to go to 'special coordinates' where one of the X^A and one of the Z^I, say X^0 and Z^0, is scaled to one. This arbitrarily singles out one of each of the basis elements namely ω_0 and α_0, and the dual $\tilde{\omega}^0$ and β^0.

For $p^A = 0$ the conditions (5.27) precisely correspond to the conditions imposed in ref. [11] with e_A being the mirror dual of the electric fluxes. Note that in ref. [11] it was assumed that all the basis forms were of pure degree and hence p^A was necessarily zero. The generalisation here is that we allow the basis forms to be of mixed degree. The next step is to show that the p^A in (5.27) corresponds to the mirror dual of the magnetic fluxes. We do not compute the entire effective action but instead only focus on the mass terms of the antisymmetric tensor, the covariant derivatives of the scalars and the Killing prepotential. Let us discuss these in turn.

The ten-dimensional type IIA action contains terms of the form $|G_{2p}|^2$ where $G_{2p} = \mathrm{d}A_{2p-1}$ is the $2p$-form field strength of the $(2p-1)$-form gauge potential A_{2p-1}. In the compactified theory the combination $\mathrm{d}A_2^- + \mathrm{d}_4 A_1^-$ appears where now d denotes the exterior derivative on M^6 while d_4 is the exterior derivative on $M^{3,1}$. Using (5.24) and (5.27) we find

$$\mathrm{d}A_2^- + \mathrm{d}_4 A_1^+ = D_2^A \omega_A + \tilde{D}_{2A} \tilde{\omega}^A \,, \qquad (5.28)$$

where

$$\begin{aligned} D_2^A &= \tilde{C}_2^0 \, p^A + \mathrm{d}_4 A_1^A \,, \\ \tilde{D}_{2A} &= \tilde{C}_2^0 \, e_A + \mathrm{d}_4 \tilde{A}_1^A \,. \end{aligned} \qquad (5.29)$$

D_2^A is invariant under the combined gauge transformations

$$\delta \tilde{C}_2^0 = \mathrm{d}_4 \Theta_1 \,, \qquad \delta A_1^A = -p^A \Theta_1 \,, \qquad (5.30)$$

where Θ_1 is a one-form gauge parameter. We see that by an appropriate gauge choice one linear combination of vectors A_1^A can be removed from the spectrum or in other words they become the longitudinal degree of freedom of a massive $\tilde{C}_2^0$. Indeed, repeating the analysis of ref. [19, 11] one easily shows that the effective action contains terms proportional to $D_2 \wedge D_2$ and $D_2 \wedge *D_2$. From this we conclude that for $p^A \neq 0$ the antisymmetric tensor $\tilde{C}_2^0$ aquires a mass by a Stueckelberg mechanism or in other words by 'eating' a vector. This is precisley what one finds in Calabi-Yau compactifications of type IIB with magnetic fluxes as computed in ref. [11] and thus we have a first crucial check that we have succesfully identified the mirror dual compactification.

---

[13]Note that $\mathrm{d}^2 = 0$ is automatically satisfied.

As a second check let us compute the Killing prepotential on the finite subspaces $U^\pm$. Using (5.20), (5.24), (5.27) and $G^+ = \mathrm{d}A_0^-$ we obtain from (4.18)

$$
\begin{aligned}
\mathcal{P}^1 + \mathrm{i}\mathcal{P}^2 &= -2\mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}} \int_{M^6} \langle \Phi^+, \mathrm{d}\Phi^- \rangle = -2\mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}} \left(X^A e_A + F_A p^A \right), \\
\mathcal{P}^3 &= \mathrm{e}^{2\phi^{(4)}} \int_{M^6} \langle \Phi^+, G^+ \rangle = \mathrm{e}^{2\phi^{(4)}} \xi^0 \left(X^A e_A + F_A p^A \right) .
\end{aligned}
\tag{5.31}
$$

These are precisely the correct Killing prepotential for the mirror dual compactification as can be seen by comparing with eq. (5.17). Under the exchange $X^A \leftrightarrow Z^I$, $F_A \leftrightarrow \mathcal{F}_I$, $e_I \leftrightarrow e_A$, $m^I \leftrightarrow -p^A$ the expressions are identical.

For completeness let us also display the results for type IIB compactifications. In this case no antisymmetric tensor becomes massive and thus it is more convenient to use the scalars in A_0^+ of (5.24) in our discussion. From the ten-dimensional type IIB action one obtains the combination $\mathrm{d}A_1^+ + \mathrm{d}_4 A_0^+$ in the four-dimensional effective action. Using (5.24) and (5.27) we find

$$
\mathrm{d}A_1^- + \mathrm{d}_4 A_0^+ = D\xi^A \omega_A + D\tilde{\xi}_A \tilde{\omega}^A
\tag{5.32}
$$

where

$$
D\xi^A = \mathrm{d}_4 \xi^A - p^A A_\mu^0, \qquad D\tilde{\xi}_A = \mathrm{d}_4 \tilde{\xi}_A - e_A A_\mu^0 .
\tag{5.33}
$$

We see that, depending on the choice of p^A, e_A, a linear combination of $\xi^A, \tilde{\xi}_A$ becomes the longitudinal degree of freedom of a massive vector A_μ^0. Again, this is precisley what one finds in Calabi-Yau compactifications of type IIA with electric and magnetic fluxes as computed in ref. [19]. The corresponding Killing prepotentials are given by

$$
\begin{aligned}
\mathcal{P}^1 - \mathrm{i}\mathcal{P}^2 &= -2\mathrm{e}^{\frac{1}{2}K_+ + \phi^{(4)}} \int_{M^6} \langle \Phi^-, \mathrm{d}\Phi^+ \rangle = -2\mathrm{e}^{\frac{1}{2}K_+ + \phi^{(4)}} \left(X^A e_A + F_A p^A \right), \\
\mathcal{P}^3 &= \mathrm{e}^{2\phi^{(4)}} \int_{M^6} \langle \Phi^-, G^- \rangle = \mathrm{e}^{2\phi^{(4)}} \left(\xi^A e_A - \tilde{\xi}_A p^A \right),
\end{aligned}
\tag{5.34}
$$

which again are perfectly mirror symmetric to (5.16).

Let us summarize. By considering compactifications of type IIA on a specific class of manifolds with $SU(3) \times SU(3)$ we were able to identify mirror duals of type IIB compactifications on Calabi-Yau threefolds with generic background H_3-flux. The dual manifolds are characterized by the condition (5.27) which generalize the half-flat conditions of ref. [11]. The new ingredient is a non-zero parameter p^A which plays the role of a dual magnetic flux.[14] Note that the quantization of the dual H_3 implies that e_A and p^A are similarly integral. In simple examples, these conditions are necessarily satisfied since e_A and p^A are related to topological invariants of the manifold.

Instead of giving (5.27) we can equally well specify differential constraints of $\Phi^\pm$. Using (5.20) one obtains

$$
\mathrm{d}\Phi^+ = -(X^A e_A + F_A p^A)\beta^0, \qquad \mathrm{d}\Phi^- = p^A \omega_A + e_A \tilde{\omega}^A .
\tag{5.35}
$$

---

[14]This dual background has also been confirmed by identifying mirror symmetric $N = 1$ domain wall solutions [63].

This compares with

$$\mathrm{d}\Phi^+ = m^I\alpha_I - e_I\beta^I, \qquad \mathrm{d}\Phi^- = (Z^Ie_I - \mathcal{F}_Im^I)\,\tilde{\omega}^0, \tag{5.36}$$

for the case of a Calabi-Yau compactification with H_3 flux. As expected we see that mirror symmetry is just exchanging odd and even forms. Note that the right hand side of $d\Phi^-$ in (5.35) is real and thus we have

$$\mathrm{d}\,\mathrm{Im}\,\Phi^- = 0. \tag{5.37}$$

The same constraint holds for half-flat manifolds but in that case also $J \wedge J$ is closed. Here, this second constraint no longer holds. Furthermore, since α_I and β^I are generically of mixed degree, Φ^- is no longer purely a three-form.

6. Generic $SU(3) \times SU(3)$ compactifications

In the previous section we considered manifolds with $SU(3) \times SU(3)$ structure which can serve as mirror dual compactifications of Calabi-Yau backgrounds with generic NS-flux. In this section we consider a more general class of compactifications by relaxing (5.27) and (5.35). As before we consider a generic truncation (5.18), with the triplets projected out, but now allow for the most general differential conditions which can be imposed on the two symplectic basis. They read

$$\begin{aligned}
\mathrm{d}\alpha_I &\sim p_I^A\omega_A + e_{IA}\tilde{\omega}^A, & \mathrm{d}\beta^I &\sim q^{IA}\omega_A + m_A^I\tilde{\omega}^A, \\
\mathrm{d}\omega_A &\sim m_A^I\alpha_I - e_{IA}\beta^I, & \mathrm{d}\tilde{\omega}^A &\sim -q^{IA}\alpha_I + p_I^A\beta^I,
\end{aligned} \tag{6.1}$$

where $p_I^A, e_{IA}, q^{IA}, m_A^I$ are four $(b^++1) \times (b^-+1)$-dimensional constant matrices. Following the discussion of the previous section, we expect these matrices to take integer values. In order to make the symplectic structure manifest let us introduce a notation for the two symplectic basis

$$\Sigma^+ := \begin{pmatrix} \omega_A \\ \tilde{\omega}^B \end{pmatrix}, \qquad \Sigma^- := \begin{pmatrix} \alpha_I \\ \beta^J \end{pmatrix}. \tag{6.2}$$

In terms of Σ^+ and Σ^- eq. (6.1) turns into

$$\mathrm{d}\Sigma^- \sim \mathcal{Q}\Sigma^+, \qquad \mathrm{d}\Sigma^+ \sim \mathcal{S}_+\mathcal{Q}^T(\mathcal{S}_-)^{-1}\Sigma^- \tag{6.3}$$

where

$$\mathcal{Q} = \begin{pmatrix} p_I{}^A & e_{IB} \\ q^{JA} & m^J{}_B \end{pmatrix}, \tag{6.4}$$

and $\mathcal{S}_+$ and $\mathcal{S}_-$ are the symplectic structures on U^+ and U^-. Note that $\mathrm{d}\Sigma^-$ and $\mathrm{d}\Sigma^+$ have to depend on the same matrix $\mathcal{Q}$ in order to ensure consistency of $\int_{M^6}\langle\Sigma^+, \mathrm{d}\Sigma^-\rangle = \int_{M^6}\langle\mathrm{d}\Sigma^+, \Sigma^-\rangle$. Furthermore $\mathrm{d}^2 = 0$ implies two additional quadratic constraints

$$\mathcal{Q}\mathcal{S}_+\mathcal{Q}^T = 0 = \mathcal{Q}^T(\mathcal{S}_-)^{-1}\mathcal{Q}, \tag{6.5}$$

– 23 –

or explicitly

$$q^{IA}m_A^J - m_A^I q^{AJ} = 0\,, \qquad p_I^A e_{AJ} - e_{IA}p_J^A = 0\,, \qquad p_I^A m_A^J - e_{IA}q^{AJ} = 0\,,$$
$$q^{AI}p_I^B - p_I^A q^{IB} = 0\,, \qquad m_A^I e_{IB} - e_{AI}m_B^I = 0\,, \qquad m_A^I p_I^B - e_{AI}q^{IB} = 0\,. \tag{6.6}$$

The 'doubly symplectic' charge matrix $\mathcal{Q}$ has also been discussed in refs. [64, 65].

Note that we can count the number of independent charges in $\mathcal{Q}$ as follows. Formally $\mathcal{Q}$ is a linear map $\mathcal{Q}: U^- \to U^+$, or equivalently $\mathcal{Q} \in (U^-)^* \otimes U^+$. The conditions (6.5) imply that images of $\mathcal{Q}$ and $\mathcal{Q}^T$ are isotropic subspaces, denoted by $L^+ := \mathrm{im}\,\mathcal{Q} \subset U^+$ and $\bar{L}^- := \mathrm{im}\,\mathcal{Q}^T \subset (U^-)^*$ respectively. Equivalently, $\mathcal{Q} \in \bar{L}^- \otimes L^+$, with, as for any linear map, $p := \dim L^+ = \dim \bar{L}^-$. Since L^+ and $\bar{L}^-$ are isotropic we have $p \le b^+ + 1$ and $p \le b^- + 1$. Furthermore, a p-dimensional isotropic subspace in a $2d$-dimensional symplectic space is determined by $2dp - \frac{1}{2}p(p-1)$ parameters. Thus counting first the parameters in choosing L^+ and $\bar{L}^-$ and then the p^2 independent elements of $\mathcal{Q}$ given L^+ and $\bar{L}^-$, we find that generically

$$\dim \mathcal{Q} = \begin{cases} (2b^- + 3)(b^+ + 1) & \text{if } b^+ \le b^- \\ (2b^+ + 3)(b^- + 1) & \text{if } b^- \le b^+ \end{cases} \tag{6.7}$$

corresponding to $p = b^+ + 1$ and $p = b^- + 1$ respectively.

The next step is to compute again the Killing prepotentials. In the type IIA low energy effective action the quantity $\mathrm{d}A_2^- + \mathrm{d}_4 A_1^+$ appears exactly as in the previous section and it again obeys the expansion (5.28). However due to (6.1) the coefficients of this expansion now read

$$D_2^A = \tilde{C}_2^I p_I^A + C_{2I} q^{AI} + \mathrm{d}_4 A_1^A\,,$$
$$\tilde{D}_{2A} = \tilde{C}_2^I e_{AI} + C_{2I} m_A^I + \mathrm{d}_4 \tilde{A}_1^A\,. \tag{6.8}$$

Recall that $\dim(\mathrm{im}\,\mathcal{Q}) = p$ with $p \le b^+ + 1$ and $p \le b^- + 1$. Hence the number of linearly independent massive antisymmetric tensors D_2^A and $\tilde{D}_{2A}$ in (6.8) is p. Thus if $b^+ \ge b^-$ at most $b^- + 1$ tensors are massive, and if $b^- \ge b^+$ at most $b^+ + 1$ tensors are massive.

The Killing prepotentials are always expressed in terms of the scalar fields. They can be computed exactly as in the previous section but now using (6.1) instead of (5.27). This yields

$$\begin{aligned} \mathcal{P}^1 + i\mathcal{P}^2 &= -2\mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}} \int_{M^6} \langle \Phi^+, \mathrm{d}\Phi^- \rangle \\ &= 2\mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}} \left(V^{-T} \mathcal{S}_- \mathcal{Q} V^+ \right) \\ &= 2\mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}} \left(-X^A e_{AI} Z^I + X^A m_A^I \mathcal{F}_I - F_A p_I^A Z^I + F_A q^{AI} \mathcal{F}_I \right), \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} \mathcal{P}^3 &= \mathrm{e}^{2\phi^{(4)}} \int_{M^6} \langle \Phi^+, G^+ \rangle \\ &= \mathrm{e}^{2\phi^{(4)}} \left(V_\xi^{-T} \mathcal{S}_- \mathcal{Q} V^+ + V_{RR}^{+T} \mathcal{S}_+ V^+ \right) \\ &= \mathrm{e}^{2\phi^{(4)}} \left[\left(X^A (\tilde{G}_{RR\,A} + e_{AI}\xi^I + m_A^I \tilde{\xi}_I) + F_A (G_{RR}^A + p_I^A \xi^I + q^{AI} \tilde{\xi}_I) \right] \,, \end{aligned} \tag{6.10}$$

where we introduced the symplectic sections

$$V^+ = \begin{pmatrix} F_A \\ X^B \end{pmatrix}, \qquad V^- = \begin{pmatrix} \mathcal{F}_I \\ Z^J \end{pmatrix}, \qquad V^-_\xi = \begin{pmatrix} \tilde{\xi}_I \\ -\xi^J \end{pmatrix}, \qquad V^+_{RR} = \begin{pmatrix} \tilde{G}_{RR\,A} \\ -G^B_{RR} \end{pmatrix},$$
(6.11)

and expanded

$$G^+ = G^A_{RR}\omega_A + \tilde{G}_{RR\,A}\tilde{\omega}^A + dA^-_0 .$$
(6.12)

Here $G^A_{RR}, \tilde{G}_{RR\,A}$ denote the RR-fluxes.[15] Note that $\mathcal{P}^1 + i\mathcal{P}^2$ has the same form as the superpotential introduced in ref. [64] where it was inferred from F-theory considerations. It would be interesting to make the correspondence with the results of ref. [64] more precise.

In the large volume limit the holomorphic prepotential F is a cubic function of the scalar fields in the vector multiplets. From (6.9) we see that the matrices p^A_I and q^{AI} multiply quadratic and cubic terms while e_{AI} and m^I_A multiply constant and linear terms. Mirror symmetry implies that there is a limit where $\mathcal{F}$ has a similar expansion. In the next section we discuss the specific example of flux backgrounds on twisted toroidal compactification in more detail, hence establishing the relation of these results with those of ref. [25].

Let us turn to type IIB. In order to see massive tensors occuring one considers the quantity $dA^+_2 + d_4C^-_1$ instead of $dA^-_1 + d_4C^+_0$ as done in (5.32). Using (6.1) and (5.25) one finds

$$dA^+_2 + d_4A^-_1 = D^I_2\,\alpha_I + \tilde{D}_{2J}\,\beta^J ,$$
(6.13)

where

$$\begin{aligned} D^I_2 &= -\tilde{C}^A_2 m^I_A + C_{2A}q^{AI} + d_4A^I_1 , \\ \tilde{D}_{2I} &= \tilde{C}^A_2 e_{AI} - C_{2A}p^A_I + d_4\tilde{A}^I_1 . \end{aligned}$$
(6.14)

The Killing prepotentials are again expressed in terms of scalar fields. Repeating the calculation of the last section with (5.27) replaced by (6.1) one finds

$$\begin{aligned} \mathcal{P}^1 - i\mathcal{P}^2 &= -2e^{\frac{1}{2}K_+ +\phi^{(4)}} \int_{M^6} \langle \Phi^-, d\Phi^+ \rangle \\ &= 2e^{\frac{1}{2}K_+ +\phi^{(4)}}\left(V^{-T}\mathcal{S}_-\mathcal{Q}V^+\right) \\ &= 2e^{\frac{1}{2}K_+ +\phi^{(4)}}\left(-Z^I e_{IA}X^A - Z^I p^A_I F_A + \mathcal{F}_I m^I_A X^A + \mathcal{F}_I q^{IA} F_A\right), \end{aligned}$$
(6.15)

and

$$\begin{aligned} \mathcal{P}^3 &= -e^{2\phi^{(4)}} \int_{M^6} \langle \Phi^-, G^- \rangle \\ &= -e^{2\phi^{(4)}}\left(V^{-T}\mathcal{S}_-\mathcal{Q}V^+_\xi + V^{-T}_{RR}\mathcal{S}_-V^-\right) \\ &= -e^{2\phi^{(4)}}\left[Z^I(\tilde{G}_{RR\,I} - e_{IB}\xi^B + p^A_I\tilde{\xi}_A) + \mathcal{F}_I(G^I_{RR} + m^I_B\xi^B - q^{IA}\tilde{\xi}_A)\right], \end{aligned}$$
(6.16)

---

[15]Note that combinations of scalars $(\xi^I, \tilde{\xi}_I)$ which is dual to the massive tensors given by (6.8) precisely drops out of the expression for $\mathcal{P}^3$ as is required for consistency. Alternatively one can formulate the supergravity in a redundant form where both scalar degrees of freedom together with antisymmetric tensors are kept [62].

	IIA	IIB
e_{AI}	massive A_μ^A	massive A_μ^I
m_A^I	massive A_μ^A	massive $\tilde{C}_2^A$
p_I^A	massive $\tilde{C}_2^I$	massive A_μ^I
q^{AI}	massive C_{2I}	massive C_{2A}

Table 4: Physical effect of different charges.

where

$$G^- = G_{RR}^I \alpha_I + \tilde{G}_{RR\,J}\beta^J + \mathrm{d}A_0^+ \,, \qquad V_\xi^+ = \begin{pmatrix} \tilde{\xi}_A \\ -\xi^B \end{pmatrix} \,, \qquad V_{RR}^- = \begin{pmatrix} \tilde{G}_{RR\,I} \\ -G_{RR}^J \end{pmatrix} \,, \quad (6.17)$$

and $G_{RR}^I, \tilde{G}_{RR\,I}$ again denote the RR-fluxes.

Let summarize the role the different $\mathcal{Q}$-charges take in the low energy effective theory. Generically they always give a mass to some of the light modes. Depending on which charge is under consideration in which type II theory either a set of vector fields or a set of antisymmetric tensor naturally becomes massive. The different cases are summarized in table 4. Of course it is always possible to rotate to a symplectic basis where all massive modes are either vectors or tensors. The most appropriate formulation of the supergravity which occurs as the low-energy effective theory for the case at hand is the one given in ref. [62]. Here all vectors and tensors are kept simultaneously and the symplectic covariance of the theory becomes manifest. A reformulation of the results obtained here in terms of the formalism of [62] will be presented elsewhere.

Finally we come to the issue of mirror symmetry. Comparing tables 2 and 3 results in a condition purely on the light spectrum. First of all the dimensions of the finite subspaces defined in (5.18) have to agree on a mirror pair of six-manifolds $(M^6, \tilde{M}^6)$ or in other words $b^+(M^6) = b^-(\tilde{M}^6)$ and vice versa. Furthermore the kinetic terms in the Lagrangian have to coincide. Here we only computed explicitly the Kähler potential of the two Kähler geometries in (5.23). We see that mirror symmetry requires the identification [12]

$$\Phi^+(M^6) \leftrightarrow \Phi^-(\tilde{M}^6)\,, \qquad \Phi^-(M^6) \leftrightarrow \Phi^+(\tilde{M}^6)\,, \tag{6.18}$$

or equivalently the exchange

$$X^A \leftrightarrow Z^I\,, \qquad F_A \leftrightarrow \mathcal{F}_I \,. \tag{6.19}$$

Comparing also the kinetic terms for the RR scalars is straightforward and results in the identification

$$\xi^A \leftrightarrow \xi^I\,, \qquad \tilde{\xi}^A \leftrightarrow \tilde{\xi}^I \,. \tag{6.20}$$

Finally comparing the Killing prepotentials (6.9), (6.10) with (6.15), (6.16) requires an identification of the charges

$$e_{AI} \leftrightarrow e_{IA}\,, \qquad q^{AI} \leftrightarrow q^{IA}\,, \qquad m_A^I \leftrightarrow -p_I^A\,, \tag{6.21}$$

and the RR-fluxes

$$G^A_{RR} \leftrightarrow -G^I_{RR}, \qquad \tilde{G}_{RR\,A} \leftrightarrow -\tilde{G}_{RR\,I} \,. \tag{6.22}$$

Thus we see that within the class of compactifications on manifolds with $SU(3) \times SU(3)$ structure mirror symmetry can be realized.

The final task of this paper is to ask to what extend the compactifications just discussed correspond to *bona fide* geometrical backgrounds. In particular, can one always find geometries with truncations satisfying (6.3), and, if not, how does this connect to the discussion in the recent literature.

7. Non-geometric backgrounds

In our discussion thus far, we have simply assumed that there are suitable $SU(3) \times SU(3)$ manifolds with truncations satisfying the differential conditions (5.27) in the case of the dual of H_3-flux, or, more generally, conditions (6.1). In the following, we will argue that this is generically not the case. Instead, following recent ideas generalizing the notion of a string background, one must consider "non-geometrical" compactifications [23]–[44].

The classic examples [27, 28, 26] of such backgrounds are tori, and orbifolds thereof, with NS three-form fluxes and the corresponding backgrounds related by successive T-duality transformations. Some of these backgrounds were shown to be non-geometric [24]. The corresponding effective theories were discussed in [33, 25]. In refs. [23] it was argued that these backgrounds correspond to non-commutative (and non-associative) geometries. The relation between these different view points has recently been clarified in ref. [42]. Note also that essentially two types of non-geometrical backgrounds have been identified: those which are locally geometrical but have no sensible global geometrical description; and those which are not even locally geometrical [35, 40]. Specific examples of the former type can be realised using the concept of a T-fold, introduced in ref. [24]. These backgrounds locally look like manifolds but the transition functions between local patches are generalised to include T-duality transformations.

Let us first give a suggestive argument as to why geometrical compactifications are not sufficient to realize all the charges in $\mathcal{Q}$. Suppose for this discussion that the relations (6.3) are exact and not up to terms which vanish under the symplectic pairing (5.19). Given that the exterior derivative maps p-forms to $(p+1)$-forms, we find that, whatever truncation we choose, the charge matrix $\mathcal{Q}$ defined in (6.3) cannot be completely generic. This suggests that in order to generate all the allowed elements in $\mathcal{Q}$ one must consider non-geometrical compactifications. The argument is a follows. Recall that $\Phi^\pm$ are expanded in terms of truncation bases Σ^+ and Σ^- as in (5.20). From (2.21) we see that, whenever $c_\parallel \neq 0$, the structure Φ^+ contains a scalar. This implies that at least one of the forms in the basis Σ^+ contains a scalar. Let us call this element Σ^+_1, and take the simple case where the only non-zero elements of $\mathcal{Q}$ are those of the form $\mathcal{Q}_{\hat{I}}{}^1$ (where $\hat{I} = 1, \ldots, 2b^- + 2$). Thus $\mathrm{d}\Sigma^-_{\hat{I}} = \mathcal{Q}_{\hat{I}}{}^1 \Sigma^+_1$ and so if $\mathcal{Q}_{\hat{I}}{}^1 \neq 0$ then $\mathrm{d}\Sigma^-_{\hat{I}}$ contains a scalar. But this is not possible if d is an honest exterior derivative, acting as $\mathrm{d} : \Lambda^p \to \Lambda^{p+1}$. The same is true if $c_\parallel$ in (2.21) is zero. In this case, there may be no scalars in any of the even forms Σ^+, and for an "honest"

d operator, there should be then no one-forms in dΣ^+. But we again see from (2.21) that Φ^- contains a one-form, and as a consequence so do some of the elements in Σ^-.

One way to generate a completely general charge matrix $\mathcal{Q}$ in this picture is to consider a modified operator d which is now a generic map d : $U^+ \to U^-$ which satisfies d$^2 = 0$ but does not transform the degree of a form properly. In particular it can map a p-form to a $(p-1)$-form. Of course, d does not act this way in conventional geometrical compactifications. One is thus led to conjecture that to obtain a generic $\mathcal{Q}$ we must consider non-geometrical compactifications. One can still use the structures (6.3) to derive sensible effective actions, expanding in bases Σ^+ and Σ^- with a generalised d operator, but there is of course now no interpretation in terms of differential forms and the exterior derivative.

As a concrete simplified example of the general ideas discussed above we consider the case of a reduction on T^6 with H_3-flux and the related twisted tori and T-dual compactifications, following [27, 28, 26, 24, 33]. Collectively we refer to such compactifications as "generalised twisted tori". We will introduce $SU(3) \times SU(3)$ structures on classes of these backgrounds and calculate the corresponding charge matrices $\mathcal{Q}$. More generally, refs. [25] (see also [40, 66]) looked at $N = 1$ orientifolds of such backgrounds, calculating the corresponding effective superpotentials. In this subsection we will review the structure of these generalised T^6 reductions. In the following subsection we calculate the corresponding $\mathcal{Q}$ matrices for our putative $SU(3) \times SU(3)$ structures and finally in the last subsection we compare with the superpotential of ref. [25].

7.1 Generalised twisted tori

A Calabi-Yau manifold in the SYZ limit can be viewed as a three-torus T^3 fibred over some base manifold [67]. In this limit mirror symmetry acts as T-duality on the T^3 fibre while leaving the base unchanged. With this prescription one can explicitly construct the mirror duals of a Calabi-Yau manifold with three-form flux H. The T^6 examples we discuss here are the trivial case of such a construction.

Let us start with a T^6 compactification where e^a are a set of one-forms defining the torus and where we include NS flux $H = \frac{1}{6} H_{abc} e^a \wedge e^b \wedge e^c$. The action of T-duality in this background has been considered by many authors. Heuristically, following the notation of ref. [25], it can be represented as follows

$$H_{abc} \xleftrightarrow{T_a} f^a{}_{bc} \xleftrightarrow{T_b} Q^{ab}{}_c \xleftrightarrow{T_c} R^{abc} \,. \tag{7.1}$$

In the SYZ formulation the different terms in (7.1) correspond to the situation where H has one, two or three 'legs' on the T^3-fibre. An H with one leg on the fibre corresponds to electric NS-fluxes and has already been considered in [11]. This leads to a geometry described by the parameters $f^a{}_{bc}$, and no H-flux. Geometrically we have a twisted torus. This is a parallelisable manifold spanned by one-forms e^a, which are now not closed, but satisfy instead

$$de^a = f^a{}_{bc}\, e^b \wedge e^c \,, \tag{7.2}$$

with $f^a{}_{bc}$ constant. Specifically, suppose only one element of H_{abc} is non-zero, and has only one leg on the T^3 fibration. After three T-dualities, we get a new manifold which is

a non-trivial T^3 fibration. The non-trivial part is a S^1 fibration over T^2, where the S^1 is the T-dual of the fibre direction along which H was non-zero.

Now suppose H has two legs along the T^3 fibration. One can again explicitly perform a local T-duality leading to a background with non-trivial geometry and H-flux. However, this cannot be done globally: there is no good global splitting between metric and B-field. Instead, one can interpret the non-trivial part of the compactification as a T^2 bundle over S^1 where there is mondromy that mixes the B-field and metric of the T^2: the bundle is being patched by an element of T-duality. As such it is a T-fold and is non-geometric. Nonetheless, the reduction can be characterized by a set of parameters Q which are related to the local metric and B-field.

Finally, the last step in the chain (7.1) is purely conjectural, since the metric does not have the isometry to perform such T-duality, and therefore the Buscher rules cannot be applied. It corresponds to an H-flux with all three legs on the fibre. In this case, [35] argues that there is not even a good local description of the geometry, though it does make sense as a conformal field theory. One way [25] to see that space-time points might not be well defined, is to note that the mirrors of D0-branes probes would be D3-branes wrapping a T^3 fibre with NS flux on the world-volume and these do not have simple moduli spaces because of the problem of satisfying the Bianchi identity $dF = H_3$. In this sense, the parameters R have no geometrical interpretation. Note that by an abuse of nomenclature, we will often refer to all the parameters H, f, Q and R as generalised "fluxes".

There are various ways to view what is encoded in these generalised fluxes. In terms of the corresponding low-energy effective theory they are related to the gauge algebra of the vector fields, coming from the symmetries of the backgrounds. One finds [68, 32, 33, 27, 24, 35, 25]

$$
\begin{aligned}
[v_a, v_b] =& H_{abc}X^c + f^c{}_{ab}v_c, \\
[v_a, X^b] =& -f^b{}_{ac}X^c + Q^{bc}{}_a v_c, \\
[X^a, X^b] =& Q^{ab}{}_c X^c + R^{abc}v_c,
\end{aligned}
\tag{7.3}
$$

where in the case of a geometrical compactification ($Q = R = 0$) the v_a generators come from the Killing vector symmetries, while X^a are associated with gauge transformations of B. Note that the algebra of diffeomorphisms parametrized by vectors and gauge transformations parametrized by one-forms is essentially the same as the Courant bracket.[16] From this perspective, in the geometrical case, one can view (7.3) as the Courant bracket algebra of Killing vectors and gauge transformations. Since, for instance, the gauge transformation of B are Abelian, one can see that the Q and R fluxes cannot arise in any convention geometrical way. Note that the Jacobi identities for the algebra then put constraints on fluxes.

An alternative picture is that the corresponding generalised geometry can be written in terms of a basis V^A of $O(6,6)$ vectors, just as for a twisted torus there is a basis of left-invariant one-forms e^a, or equivalently vectors $\tilde{v}_a$. Just as the structure constants $f^a{}_{bc}$ appear in the Lie algebra of the $\tilde{v}_a$, so the generalised fluxes appear in the Courant

---

[16]The Courant bracket between two elements $x + \xi$ and $y + \eta$ in E is given by $[x + \xi, y + \eta] = [x, y] + \mathcal{L}_x \eta - \mathcal{L}_y \xi - \frac{1}{2}d(i_x \eta - i_y \xi)$ where $[x, y]$ is the usual Lie bracket of vector fields and $\mathcal{L}_x$ is the Lie derivative.

bracket algebra of the V^A. Note that this is a complementary picture to the one just given: on a twisted torus the right-invariant vector fields v_a generate the isometries, while the left-invariant vector fields $\tilde{v}_a$ are used to define the metric.

A third picture, useful when relating to $SU(3) \times SU(3)$ structures is to ask how the fluxes enter the exterior algebra of the forms. For a geometrical background it is natural to consider forms of the type $\omega = e^{-B}\omega_{m_1...m_p}e^{m_1} \wedge \cdots \wedge e^{m_p}$ with $\omega_{m_1...m_p}$ constant. We include the twisting by B so that ω is an element of the generalised spinor bundle $S(E)$. Acting with d on ω we find

$$\mathrm{d}\omega = -H \wedge \omega + f \cdot \omega \tag{7.4}$$

where $(f \cdot \omega)_{m_1...m_{p+1}} = f^a{}_{[m_1 m_2|}\omega_{a|m_3...m_{p+1}]}$. The natural non-geometrical extension is then to an operator $\mathcal{D}$ such that [25]

$$\mathcal{D}\omega := -H \wedge \omega + f \cdot \omega + Q \cdot \omega + R\llcorner\omega, \tag{7.5}$$

where $Q\cdot$ and $R\llcorner$ are defined by

$$(Q \cdot \omega)_{m_1...m_{p-1}} = Q^{ab}{}_{[m_1}\omega_{|ab|m_2...m_{p-1}]}, \quad (R\llcorner\omega)_{m_1...m_{p-3}} = R^{abc}\omega_{abcm_1...m_{p-3}} . \tag{7.6}$$

Requiring $\mathcal{D}^2 = 0$ implies that same conditions on fluxes as arose from the Jacobi identities for (7.3). The connection $\mathcal{D}$ appears in the Bianchi identities for the RR fluxes, which in the presence of geometric and non-geometric fluxes read $\mathcal{D}F = 0$. Note that in our analysis the equality in (7.5) will be relaxed to an equivalence up to terms vanishing under the symplectic pairing (5.19).

7.2 Generalised twisted tori and $SU(3) \times SU(3)$ structures

We will now try and relate the fluxes (7.1) in the generalised twisted tori examples to our generic $SU(3) \times SU(3)$ reductions discussed in section 6. This will allow us to see how the charges $\mathcal{Q}$ can be realised in terms of the fluxes and hence, in this particular example, which terms in $\mathcal{Q}$ come from conventional compactifications and which from non-geometrical backgrounds.

Let us consider first an $SU(3)$ structure on the generalised twisted torus manifold. In the geometrical case, the manifold is parallelisable and there is non-trivial H-flux. To define the $SU(3)$ structure we introduce three complex one-forms e^i (with conjugates $\bar{e}^{\bar{i}}$). In order to keep the discussion tractable we will assume that there is $\mathbb{Z}_3$ symmetry under permutation of the three e^i. In the simple case where the manifold is T^6 this implies that we are considering the product $T^2 \times T^2 \times T^2$ and assuming the metric and H-field are the same on each T^2.

In terms of $SU(3)$ structure this means we fix identical complex structures and Kähler forms on each T^2 (or rather in terms of each e^i). There are then two moduli: the complex Kähler modulus t and complex structure τ of each T^2. We thus have, as in section 5.1

$$\Phi^+ = e^{-B^{\mathrm{fl}}}e^{\mathrm{i}t\lambda}, \qquad \Phi^- = e^{-B^{\mathrm{fl}}}\Omega_\tau^1 \wedge \Omega_\tau^2 \wedge \Omega_\tau^3, \tag{7.7}$$

where $\lambda = 2\mathrm{i}\delta_{i\bar{j}}e^i\bar{e}^{\bar{j}}$ and $\Omega_\tau^i = \frac{1}{2}(1+\tau)e^i + \frac{1}{2}(1-\tau)\bar{e}^{\bar{i}}$ define the complex structure on each T^2, while $\mathrm{d}B^{\mathrm{fl}} = H$. We are expanding in a basis of even forms

$$\Sigma^+ = (\omega_0, \omega_1, \tilde{\omega}^0, \tilde{\omega}^1) = e^{-B^{\mathrm{fl}}}\left(1, \tfrac{1}{6}\lambda^2, \tfrac{1}{6}\lambda^3, \lambda\right), \tag{7.8}$$

– 30 –

and of odd forms

$$\Sigma^- = (\alpha_0, \alpha_1, \beta^0, \beta^1) = \mathrm{e}^{-B^{\mathrm{fl}}} \left(\mathrm{Re}\, \Omega_3, \mathrm{Re}\, \chi_3, -\mathrm{Im}\, \Omega_3, -3\,\mathrm{Im}\, \chi_3 \right) \tag{7.9}$$

where

$$\Omega_3 = \tfrac{2}{3}\epsilon_{ijk}e^i e^j e^k \,, \qquad \chi_3 = \tfrac{4}{3}(\bar{e}^1 e^2 e^3 + \mathrm{cyclic}) =: \tfrac{2}{3}\rho^i{}_{jk}\,\delta_{i\bar{l}}\,\bar{e}^l e^j e^k \,. \tag{7.10}$$

The components of ρ satisfy $\rho^1{}_{23} = -\rho^1{}_{32} = \rho^2{}_{31} = -\rho^2{}_{13} = \rho^3{}_{12} = -\rho^3{}_{21} = 1$, with the others being zero. The forms satisfy additionally (5.21).

The fluxes (7.1) of the non-trivial geometry are encoded in the H-flux and the twisted geometry (7.2). Specifically, respecting the $\mathbb{Z}_3$ symmetry we have

$$H_3 = \mathrm{d}B^{\mathrm{fl}} = H^0 \,\mathrm{Re}\, \Omega_3 + H^1 \,\mathrm{Re}\, \chi_3 - H_0(-\mathrm{Im}\, \Omega_3) - H_1(-3\,\mathrm{Im}\, \chi_3)\,, \tag{7.11}$$

while decomposing (7.2) in terms of holomorphic and antiholomorphic indices, and imposing the $\mathbb{Z}_3$ symmetry, gives

$$\mathrm{d}e^i = \tfrac{1}{6}A\rho^i{}_{jk}\, e^j e^k + \tfrac{1}{6}B\rho^{ij}{}_k\, \delta_{j\bar{l}}\, \bar{e}^l e^k + \tfrac{1}{6}C\epsilon^{ijk}\, \delta_{j\bar{l}}\delta_{k\bar{m}}\, \bar{e}^l \bar{e}^m. \tag{7.12}$$

Using (7.11) and (7.12) to compute the exterior derivatives of the elements of Σ^+, and expressing them as linear combinations of the forms in Σ^- we obtain an expression for the charge matrix $\mathcal{Q}$ in terms of the structure constants A, B and C and the H-fluxes H^I and H_I. We get

$$\begin{aligned} H_I &= e_{I0}\,, \qquad H^I = m^I{}_0\,, \\ 6A + \bar{B} &= 3p_1{}^1 + \mathrm{i}q^{11}\,, \\ C &= \tfrac{1}{6}p_0{}^1 + \tfrac{1}{6}\mathrm{i}q^{01}\,, \end{aligned} \tag{7.13}$$

and $q^{I0} = p_I{}^0 = e_{I1} = m^I{}_1 = 0$.[17] The charge matrix is therefore

$$\mathcal{Q} = \begin{pmatrix} 0 & \mathrm{Re}\, C & H_1 & 0 \\ 0 & \tfrac{1}{18}\mathrm{Re}\, D & H_2 & 0 \\ 0 & \mathrm{Im}\, C & H^1 & 0 \\ 0 & \mathrm{Im}\, D & H^2 & 0 \end{pmatrix}\,, \tag{7.14}$$

where $D = 6A + \bar{B}$. This implies that only half of the charges are turned on via H-flux and geometric fluxes. We therefore expect the other half of the charges $\mathcal{Q}_{\hat{I}1}$, $\mathcal{Q}_{\hat{I}4}$ ($\hat{I} = 1, \dots, 4$) to correspond to non-geometric fluxes. There are as many $Q^{ab}{}_c$ fluxes respecting the $\mathbb{Z}_3$ symmetry as there are $f^a{}_{bc}$, and the same is true for R^{abc} and H_{abc}. It is reasonable to expect that turning them on would complete the entries of the charge matrix $\mathcal{Q}$. Let us show that this is indeed the case.

Let us use the operator $\mathcal{D}$ in (7.5) to define the fluxes Q and R. Replacing d in (6.1) with $\mathcal{D}$ we find that the full charge matrix is then given by

$$\mathcal{Q} = \begin{pmatrix} R_1 & \mathrm{Re}\, C & H_1 & \tfrac{2}{3}\mathrm{Im}\, \tilde{C} \\ R_2 & \tfrac{1}{18}\mathrm{Re}\, D & H_2 & \tfrac{2}{9}\mathrm{Im}\, \tilde{D} \\ R^1 & \mathrm{Im}\, C & H^1 & \tfrac{2}{3}\mathrm{Re}\, \tilde{C} \\ R^2 & \mathrm{Im}\, D & H^2 & \tfrac{1}{6}\mathrm{Re}\, \tilde{D} \end{pmatrix}\,, \tag{7.15}$$

---

[17]Note that for our choice of $SU(3)$ structure not all fluxes of (7.12) appear but only the combination $\bar{B} + 6A$.

where $\tilde{D} = \tilde{A} + \bar{\tilde{B}}$ and $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are defined in direct analogy with A, B and C, while R_I, R^I are the components of R-flux defined in analogy with (7.11). We see, as promised, that the missing half of the $\mathcal{Q}$'s are indeed given by the non-geometric fluxes Q and R. We conclude that the charge matrix $\mathcal{Q}$ represents geometric as well as non-geometric fluxes, and all of the elements of $\mathcal{Q}$ can in principle be generated by an appropriate H, f, Q or R-flux. Note that the flux parameters are not all independent but have to satisfy the constraint (6.6). The same constraint also arises from requiring $\mathcal{D}^2 = 0$. In this particular case, using the general expression (6.7), we have ten independent charges.

We can also generalize this calculation to the case of an $SU(3) \times SU(3)$ structure. From the discussions in the previous sections, we expect this setup to accommodate more of the $\mathcal{Q}$ charges in a purely geometric background. We will see that this is indeed the case.

Specifically we assume that there is an $SU(2)$ structure on the generalised T^6 again with $\mathbb{Z}_3$ symmetry. Using the same forms e^i, let us choose e^3 to be the holomorphic vector of the $SU(2)$ structure. In the language of eq. (2.21), we are taking $c_\parallel = 0$, $c_\perp = 1$ and $v + \mathrm{i}v' = e^3$. The $SU(2)$ structure is then equivalent to two $SU(3)$ structures, defined by the holomorphic vectors (e^1, e^2, e^3) and $(\hat{e}^1, \hat{e}^2, \hat{e}^3) = (\bar{e}^1, \bar{e}^2, e^3)$. The $\mathbb{Z}_3$ acts by a simultaneous permutation of (e^1, e^2, e^3) and $(\hat{e}^1, \hat{e}^2, \hat{e}^3)$. We can again find suitable bases Σ^+ and Σ^- preserving the $\mathbb{Z}_3$ symmetry and (5.19) and (5.21). The bases with the minimum number of elements are given by

$$\Sigma^+ = \mathrm{e}^{-B^{\mathrm{fl}}} \begin{pmatrix} 2\,\mathrm{Re}\,(\omega_2 + \xi_2) \\ 8\,\mathrm{Im}\,\omega_2 - 4\,\mathrm{i}\mathrm{Re}\,(\omega_2 + \xi_2)e^3\bar{e}^3 \\ -4\mathrm{i}\mathrm{Re}\,(\omega_2 - \xi_2)e^3 e^{\bar{3}} \\ -\frac{2}{3}\mathrm{Re}\,(\omega_2 - \xi_2) + \frac{2}{3}\mathrm{Im}\,\omega_2 e^3 e^{\bar{3}} \end{pmatrix}, \quad \Sigma^- = \mathrm{e}^{-B^{\mathrm{fl}}} \begin{pmatrix} 2\mathrm{Re}\,e^3 \\ -2\mathrm{Im}\,e^3 + \mathrm{Re}\,e^3 j \\ -\mathrm{Im}\,e^3 j^2 \\ -\frac{1}{3}\mathrm{Re}\,e^3 j^2 + \frac{4}{3}\mathrm{Im}\,e^3 j \end{pmatrix},$$

where wedge products are understood and where $\omega_2 = e^1 \wedge e^2$, $\chi_2 = \bar{e}^1 \wedge e^2$, and $j = 2\mathrm{i}(e^1 \wedge \bar{e}^1 + e^2 \wedge \bar{e}^2)$. Note that, with $B^{\mathrm{fl}} = 0$, there are neither scalars, nor six-forms in the basis of even forms. In addition, unlike in the $SU(3)$ case with $B^{\mathrm{fl}} = 0$, it is not possible to find a basis of forms of pure degree.

The "metric fluxes" are introduced via the exterior derivatives of the one-forms, given by (7.12). In the symmetric setup, the structure constants are again proportional to ϵ^{ijk} and $\rho^i{}_{jk}$. As before the H-flux, comes from the twisting of the basis forms by $e^{-B^{\mathrm{fl}}}$. Since there are no scalars in the basis of even forms, we should not expand H_3 in the basis of odd forms, but rather simply calculate the parameters $H_{\hat{I}\hat{A}} = \int \langle H_3 \wedge \Sigma^-_{\hat{I}}, \Sigma^+_{\hat{A}} \rangle$.

The structure constants and H-flux generate the following charge matrix

$$\mathcal{Q} = \begin{pmatrix} \frac{1}{12}\mathrm{Re}\,E^+ & \frac{1}{72}\mathrm{Im}\,F + h_i^+ & 3h_i^- & -\frac{1}{6}\mathrm{Re}\,E^- + 4h_i^0 \\ -\frac{1}{12}\mathrm{Im}\,E^+ & \frac{1}{24}\mathrm{Re}\,F + h_r^+ & \frac{1}{12}\mathrm{Im}\,E^- + 3h_r^- & \frac{1}{6}\mathrm{Im}\,(2E^- + F) + 4h_r^0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{108}\mathrm{Im}\,(3E^+ - E^-) & \frac{1}{9}\mathrm{Im}\,F & -\frac{2}{9}\mathrm{Re}\,(E^+ + F) \end{pmatrix},$$

where we have defined

$$E^\pm = A + C \pm 2B, \quad F = -A + C. \tag{7.16}$$

The parameters A, B, and C are defined in (7.12), and $h_{r,i}^{\pm,0}$ are the different H-flux charges that can be turned on. If we expanded H_3 in 20 independent three-forms, only six combinations of them would contribute to the charges. Explicitly,

$$H_3 = \tfrac{1}{6}h_r^\pm \mathrm{Re}\,(\omega_2 \pm \xi_2)\mathrm{Re}\,e^3 + \tfrac{1}{6}h_i^\pm \mathrm{Re}\,(\omega_2 \pm \xi_2)\mathrm{Im}\,e^3 + h_r^0 \mathrm{Im}\,(\omega_2)\mathrm{Re}\,e^3$$
$$+ h_i^0 \mathrm{Im}\,(\omega_2)\mathrm{Im}\,e^3 + \ldots \tag{7.17}$$

where the $+\ldots$ are pieces that do not contribute to the charge matrix. We see that in the $SU(2)$ case, 11 out of the 16 charges can be turned on via geometric fluxes, as oposed to 8/16 for the $SU(3)$ case. The remaining 5 charges can be turned on by Q- and R-fluxes. Note once more that there are (six) conditions on the charges coming from constraint (6.6). For the charge matrix (7.16), two of these are automatic, while one needs to impose the other four.

We conclude that in order to generate non-zero entries for the full charge matrix we need geometric as well as non-geometric fluxes both in the $SU(3)$ and in the $SU(2)$ case. However, in the latter the number of charges that can be turned on via geometric fluxes is generically larger than in the former.

7.3 Superpotentials

We can further support the claim that a generic $\mathcal{Q}$ contains geometric and non-geometric fluxes by computing the superpotentials (6.9) and (6.15) for a given $\mathcal{Q}$, and comparing to that of ref. [25]. Starting from IIA and IIB compactifications on the $\mathbb{Z}_3$ symmetric $T^2 \times T^2 \times T^2$ torus with an $SU(3)$-structure, flux and O6 and O3 planes respectively, the authors of [25] used T-duality arguments to propose a generic form for the superpotential valid also for dual non-geometrical compactifications. The superpotentials are functions of the dilaton S, two further $N = 1$ moduli X and Y and the fluxes H, f, Q, and R. They have the generic form

$$\mathcal{W} = P_1(X) + SP_2(X) + YP_3(X), \tag{7.18}$$

where $P_{1,2,3}(X)$ are cubic polynomials with the coefficients being the (geometric and non-geometric) NS and RR fluxes. P_1 depends on RR fluxes only, while the NS fluxes generate P_2 and P_3. Each type of flux contributes to a term with a given dependence on the moduli. For example, the term proportional to SX^2 is proportional to Q-flux in type IIA, while it corresponds to H-flux in type IIB.

Let us compare (7.18) with the superpotential obtained from the type IIA and type IIB superpotentials given in (4.19) and (4.20), for an O6 and an O3 orientifold projection respectively. The $N = 1$ supersymmetry preserved by these projections correspond to $\alpha = \pi/4$, $\beta = \pi/2$, giving

$$\mathcal{W}_{\mathrm{IIA/O6}} = \int \left\langle \Phi^+, \mathrm{d}\Pi^- \right\rangle, \qquad \Pi^- := A_0^- + \mathrm{iRe}\,(C\Phi^-), \tag{7.19}$$

$$\mathcal{W}_{\mathrm{IIB/O3}} = -\int \left\langle \Phi^-, \mathrm{d}\Pi^+ \right\rangle, \qquad \Pi^+ := A_0^+ + \mathrm{iRe}\,(e^{-\phi}\Phi^+), \tag{7.20}$$

where $A_0^\pm$ are the RR potentials defined in (3.5), (3.6) with field strength $G^\pm$ defined in (4.10). In ref. [48] it was shown that $\Pi^\pm$ are the correct $N = 1$ Kähler coordinates for

the orientifolds. C is a 'compensator' field proportional to $e^{-\phi}$ (for the precise definition see [48]).

Recall that for the symmetric $(T^2)^3$ setup, the $\Phi^\pm$ corresponding to a single $SU(3)$ are given by (7.7) with moduli t and τ. After the O6 orientifold projection t remains an $N = 1$ modulus (which is commonly called T) while τ is constrained to be real and it combines with a RR scalar ξ_1 to form the $N = 1$ modulus $U = \xi_1 + iC\tau^2$ which enters Π^-. The second variable is $S = \xi_0 + iC$. In type IIB, the O3 projection requires t to be real, and the $N = 1$ moduli are given by $U = \tau$, $T = \xi_1 + \mathrm{i}e^{-\phi}t^2$ and $S = \xi_0 + \mathrm{i}e^{-\phi}$ (see [48] for further details).

Substituting these expressions and using the bases (7.8) and (7.9) and the general expressions (6.3) and (6.4) we find

$$
\begin{aligned}
\mathcal{W}_{\mathrm{IIA/O6}} =& U\Big[\mathrm{i}(3e_{00} - e_{10}) - T(3p_0{}^1 - p_1{}^1) - 3\mathrm{i}T^2(3e_{01} - e_{11}) - T^3(3p_0{}^0 - p_1{}^0)\Big] \\
&+ S\Big[\mathrm{i}(e_{00} + e_{10}) - T(p_0{}^1 + p_1{}^1) - 3\mathrm{i}T^2(e_{01} + e_{11}) - T^3(p_0{}^0 + p_1{}^0)\Big] \, .
\end{aligned}
\tag{7.21}
$$

for type IIA, and

$$
\begin{aligned}
\mathcal{W}_{\mathrm{IIB/O3}} =& T\Big[3\mathrm{i}(e_{01} + e_{11}) + U(3m^0{}_1 + m^1{}_1) - 3\mathrm{i}U^2(3e_{01} - e_{11}) - U^3(m^0{}_1 - m^1{}_1)\Big] \\
&+ S\Big[-\mathrm{i}(e_{00} + e_{10}) + U(m^1{}_0 + 3m^0{}_0) - \mathrm{i}U^2(3e_{00} - e_{10}) + U^3(m^0{}_0 + m^1{}_0)\Big] \, .
\end{aligned}
\tag{7.22}
$$

for type IIB.

These superpotentials are symmetric under the mirror map (6.18). Furthermore, they contain all the terms in (7.18) depending on NS fluxes, namely P_2 and P_3, if we identify $X = T$ and $Y = U$ for type IIA, and $X = U$ and $Y = T$ for type IIB. The first lines of (7.21) and (7.22) correspond to the terms in P_2, while the second line to those in P_3. In the IIA expression of ref. [25], the terms with no power of T (appearing first on the first and second lines of (7.21), proportional to e_{I0}) come from H-flux. The terms linear in T come from f-fluxes and the ones quadratic in T from Q-fluxes, while the cubic ones involve the conjectured R-fluxes. This is in perfect agreement with (7.13) and (7.15), where we identified e_{I0} charges as H-flux, $p_I{}^1$ as f-flux, e_{I1} as Q-flux and $p_I{}^0$ as R-flux. Note that the fluxes m and q drop from the IIA/O6 superpotential (or more precisely, they are projected out by the orientifold projection). In type IIB with an O3 projection, all the terms containing the modulus S correpond to H-fluxes, while the ones with a T modulus are generated by Q-fluxes. (f and R fluxes are not allowed by an O3 projection.) This is again consistent with (7.13), (7.15) where $m^I{}_0$ has been identified with H^I, while $m^I{}_1$ with Q-flux.

From these examples, we conclude that the general matrix $\mathcal{Q}$ contains all possible NS fluxes. Note that the mapping between the charges (e, m, p, q) and the fluxes (H, f, Q, R) depends on the choice of basis (7.8) and (7.9). However, the fact that some of these fluxes cannot be obtained from an honest exterior derivative (or from purely geometric fluxes) is a basis independent statement.

The form of the generalised derivative (7.5) suggests that both Q and R fluxes are associated with deformations of the usual exterior algebra. However, we also know that backgrounds with non-trivial Q-fluxes are still locally geometrical. The non-geometry only appears globally. Thus one might still expect the exterior algebra to be undeformed working on a patch. A possible resolution is that (7.5) is too strong for two reasons. First it gives the action of $\mathcal{D}$ on forms of pure degree, whereas we have already seen generically we are interested in basis forms of mixed degree. Secondly, for our $SU(3) \times SU(3)$ structure we also only require an equivalence "$\sim$" up to terms which vanish under the symplectic pairing (5.19). It would be interesting to clarify if the exterior derivative actually needs to be modified to define Q given these two subtleties. For now, let us simply connect the analysis here to the discussion in [42], which will provide some evidence that such a resolution is possible.

In section 5.1 we observed that the effect of the H-flux was to twist the geometrical basis of forms so that, for instance, $\omega = \mathrm{e}^{-B^{\mathrm{fl}}} \omega^{(0)}$, which were forms of mixed degree. It is natural to ask if, for instance, the Q-charge can also be realised as a twisting of the geometrical basis, again giving forms of mixed degree. This can indeed be done, but the price to pay is higher than for H. Under two T-dualities along the B-field directions, the B-transform is mapped to a β-transform [42] (see also [69]), where β^{ab} is a bivector along the T-dualized directions. Defining a new basis $\omega = \mathrm{e}^{\beta \llcorner} \omega^{(0)}$ one would then expect that the corresponding exterior algebra encodes the Q-charges, without modifying the d operator. This is fine locally but globally the geometrical picture breaks down. Non-trivial H-flux corresponds to patching the bundle E with non-trivial transformations $B_\alpha = B_\beta + \mathrm{d}A_{\alpha\beta}$ on the intersection $U_\alpha \cap U_\beta$. The pure spinors $\Phi^\pm$ are global sections of the twisted spin bundle $S(E)$. In the case of a torus fibration with H-flux there are B-transformation monodromies on the T^3 fibre as one transverses a loop in the base. However, since $\Phi^\pm$ are global sections they are invariant under these monodromies. For the dual T^3-fibred background, the patching is by β-transformations, that is T-dualities on the T^3 fibres. Such a background is thus not globally geometrical. There are T-duality-valued monodromies, which have, for instance, the effect of changing the dimension of a brane [24, 42] and the type k of a pure spinor.[18] However, the new background still leads to a supersymmetric effective action, which means there is still a notion of a global $SU(3) \times SU(3)$ structure. In other words there is a unique pair of pure spinors $\Phi^\pm$ on each local geometrical patch. In going between patches these are related by T-duality transformations, in such a way that they are invariant under the monodromies. Expanding in terms of basis forms Σ^+ and Σ^-, this implies that each element of the basis should similarly be globally defined in this generalised "bundle" patched by T-duality. The usual exterior derivative acting on the basis elements on each local geometrical patch should encode the Q-fluxes, and the local expressions for the superpotential and so on will still hold. This is one way of suggesting why the geometrical $SU(3) \times SU(3)$ expressions give the correct low-energy effective theory in the case of non-geometrical compactifications with Q-flux.

---

[18]A pure spinor can aways be written as $e^A \theta^1 \wedge \cdots \wedge \theta^k$, where A is a complex two-form and θ^i are complex one-forms. The integer k is the "type" of the pure spinor.

In summary, we have shown that a generic matrix $\mathcal{Q}$ contains geometric as well as non-geometric NS fluxes, by calculating $\mathcal{Q}$ in terms of the fluxes H, f, Q and R in the context of generalised twisted-tori. We further show that, in the orientifold case, this then reproduces the superpotentials given in [25]. Remarkably we note that treating the exterior derivative operator in (6.1) as a generalised linear operator on the bases forms Σ^+ and Σ^- reproduces the conjectured non-geometrical superpotentials even when the background is not even locally geometrical.

8. Conclusions

In this paper we completed our study of type II compactifications on manifolds with $SU(3) \times SU(3)$ structure by further generalizing the formalism developed in ref. [45]. We first decomposed the ten-dimensional fields under $SU(3) \times SU(3)$ projecting out all representations $(\mathbf{3}, \mathbf{1})$, $(\mathbf{1}, \mathbf{3})$ and their complex conjugates. This corresponds to a reorganization of the ten-dimensional fields in terms of '$N = 2$ multiplets' without performing a Kaluza-Klein reduction. In this ten-dimensional framework we computed the equivalent of the gravitino mass matrix S_{AB} and the $N = 1$ superpotential W for type IIA and type IIB. These have the same functional expression in terms of the two pure spinors $\Phi^\pm$ and RR field strengths $G^\pm$ as their $SU(3)$ structure counterparts found in [45], and are in particular mirror symmetric under a chirality exchange of the pure spinors and RR fluxes.

We discussed the conditions for a consistent reduction where the infinite tower of Kaluza-Klein states is truncated to a set of light modes of the compactification. Such conditions arise from demanding that the local special Kähler geometry of the untruncated theory descends to the moduli space of truncated modes. (Note the question of when such truncations exist remains an open problem, see also [50].) Upon meeting these conditions, the resulting theory is a four-dimensional $N = 2$ supergravity, with generically massive antisymmetric tensors. For a specific choice of truncation, we precisely reproduced the type IIA dual of type IIB supergravity on Calabi-Yau threefolds with magnetic NS three-form fluxes. This theory was missing in [11, 45] but can be found when the compactification manifold has $SU(3) \times SU(3)$ instead of $SU(3)$ structure. The crucial new ingredient is the existence of all odd forms including one- and five-forms which are absent in $SU(3)$ structure compactifications. This allows one to generalise previous Ansätze for the exterior derivatives of the basis forms, involving a doubly symplectic charge matrix $\mathcal{Q}$, which encodes the full set of NS fluxes (three-form flux H_3 and torsion).

For general $SU(3) \times SU(3)$ structure compactifications the low-energy effective type IIA and type IIB theories are perfectly mirror symmetric under exchange of the "moduli" X^A and Z^I parameterising the bundles of even and odd pure spinors (some of these are massive and therefore not moduli in the strict sense), an exchange of the RR fluxes $G_{\mathrm{RR}\,A}$ and $G_{\mathrm{RR}\,I}$, and a symplectic transposition of the charge matrix $\mathcal{Q}$. The latter maps in particular the "magnetic" fluxes $m^I{}_A$ to the new set of fluxes $p^A{}_I$. The question of the existence of manifolds of $SU(3) \times SU(3)$ structure was not adressed in this paper. However, the restoration of mirror symmetry seems to be a strong argument in its favor.

In spite of the fact that $SU(3) \times SU(3)$ structures (or the existence of one- and five-forms in the basis of odd forms) allow one to turn on more components of $\mathcal{Q}$ than those allowed by pure $SU(3)$ structures, we showed that entirely geometric fluxes (H_3 plus torsion) do not suffice to generate all components of $\mathcal{Q}$. The extra components were shown to be associated to non-geometric fluxes, which arise in certain standard cases by performing successive T-dualities on backgrounds with purely geometric fluxes. A general charge matrix corresponds to a generic map from the truncated space of even forms to the space of odd forms. In the analysis of [25] it corresponds to a generalised nilpotent operator $\mathcal{D} = -H \wedge + f \cdot + Q \cdot + R \llcorner$ acting on the basis of forms. The nilpotency condition translates into quadratic constraints on $\mathcal{Q}$ that leave $(2b^+ + 3)(b^- + 1)$ (for $b^+ > b^-$) independent components in the charge matrix.

The non-geometrical fluxes Q are associated with a background which is locally geometrical but globally is patched using T-duality transformations. As such it can be interpreted as a "T-fold" following [24]. The non-geometrical fluxes R correspond to backgrounds which are not even locally geometrical. These have been discussed in [35]. In the former case, supersymmetry implies that one can still identify a local $SU(3) \times SU(3)$ structure. In fact, given that T-duality transformations by which the background is patched should not break supersymmetry, we would expect the $SU(3) \times SU(3)$ is globally defined, in the sense that there are no monodromies. This will not however be true of the metric and B-field, since there is no longer a global "polarization" (in the language of [24]). For instance, there are generically monodromies under which D0-branes become D2-branes and so on. Remarkably, we find that while derived using the assumption that we had a geometrical background, our expressions such as that of the superpotential seem to correctly reproduce the gaugings or masses coming from such non-geometric fluxes. The only modification is to allow a generalised exterior derivative operator or, in the truncated version, a general charge matrix $\mathcal{Q}$. While in the case of Q fluxes this might be assumed to be related to the local geometrical structure, the expressions also appear to hold for R-fluxes where the background is not even locally geometrical.

Acknowledgments

perial College for hospitality and financial support during initial and final stages of this work.

A. Spinor conventions

For convenience, in this appendix we will summarize our conventions for $O(6,6)$ spinors and identify the various relations to conventional $Spin(6)$ representations. We start by defining our conventions for $Spin(6)$ spinors.

A.1 $Spin(6)$ spinors

The Clifford algebra $\mathrm{Cliff}(6,0;\mathbb{R})$ is generated by the gamma matrices γ_m satisfying

$$\{\gamma_m, \gamma_n\} = 2g_{mn}. \tag{A.1}$$

where g is a positive definite six-dimensional metric. Let ϵ_g be an orientation compatible with g (and thus fixed up to a sign). We can define the standard intertwiners

$$\gamma_m^\dagger = A\gamma_m A^{-1}, \qquad -\gamma_m^T = C^{-1}\gamma_m C, \qquad -\gamma_m^* = D^{-1}\gamma_m D, \tag{A.2}$$

and the chirality operator $\gamma_{(6)} = \frac{1}{6!}\epsilon_g^{m_1\ldots m_6}\gamma_{m_1\ldots m_6}$. Note one can always choose a representation where $A = C = D = 1$ and the γ^m are imaginary and anti-symmetric. For a spinor θ it is useful to define

$$\bar{\theta} = \theta^\dagger A, \qquad \theta^t = \theta^T C^{-1}, \qquad \theta^c = D\theta^*. \tag{A.3}$$

We also define chiral spinors by $\gamma_{(6)}\theta_\pm = \mp i\theta_\pm$ with $\theta_\pm^c = \theta_\mp$.

A.2 $Spin(6,6)$ spinors

Let $\Pi, \Sigma, \ldots$ denote $O(6,6)$ vector indices on the generalised bundle E. (For simplicity here we will assume $E = F \oplus F^*$.) The Clifford algebra $\mathrm{Cliff}(6,6;\mathbb{R})$ is generated by the gamma matrices Γ^Σ satisfying

$$\{\Gamma_\Pi, \Gamma_\Sigma\} = 2\mathcal{G}_{\Pi\Sigma}, \tag{A.4}$$

where $\mathcal{G}$ is the $O(6,6)$ invariant metric (2.4). The $O(6,6)$ spinors $\chi_\epsilon \in S$ can be chosen to be Majorana-Weyl and we write $\chi_\epsilon^\pm \in S^\pm$ for the two chiralities. As usual one can define the intertwiner $-\Gamma_\Sigma^T = \mathcal{C}^{-1}\Gamma_\Sigma\mathcal{C}$. Using $\mathcal{C}$ one can define a spinor bilinear (which defines the Mukai pairing) by

$$\psi_\epsilon^t \cdot \chi_\epsilon := \psi_\epsilon^T \mathcal{C}^{-1}\chi_\epsilon. \tag{A.5}$$

Since $\mathcal{C}^T = -\mathcal{C}$ this is actually defines a symplectic structure. The Majorana condition uses the intertwiner $\Gamma_\Sigma^* = \tilde{\mathcal{D}}^{-1}\Gamma_\Sigma\tilde{\mathcal{D}}$, and reads $\chi_\epsilon^{\tilde{c}} := \tilde{\mathcal{D}}\chi_\epsilon^* = \chi_\epsilon$.

There are a number of different sub-groups of $O(6,6)$ under which we can decompose the spinor representation. First, the decomposition $E = F \oplus F^*$ defines a $GL(6,\mathbb{R}) \subset O(6,6)$ group. A vector $V \in E$ can then be decomposed into an ordinary vector and

one-form $V = x + \xi$. Furthermore, under this map S is isomorphic to the bundle of forms $S \simeq \Lambda^* F^*$ (or for chiral spinors $S^+ \simeq \Lambda^{\text{even}} F^*$ and $S^- \simeq \Lambda^{\text{odd}} F^*$)

$$\chi_\epsilon \sim \chi = \chi_0 + \cdots + \chi_6, \tag{A.6}$$

where $\chi_p \in \Lambda^p F^*$ and the isomorphism depends on a choice of volume form ϵ (though is independent of the sign of ϵ). In this basis, the metric $\mathcal{G}$ has the form (2.4) and we can decompose the gamma matrices as

$$V^\Sigma \Gamma_\Sigma = x^m \check{\Gamma}_m + \xi_m \hat{\Gamma}^m \tag{A.7}$$

so that (A.4) becomes

$$\{\check{\Gamma}_m, \check{\Gamma}_n\} = \{\hat{\Gamma}^m, \hat{\Gamma}^n\} = 0, \qquad \{\check{\Gamma}_m, \hat{\Gamma}^n\} = 2\delta_m^n. \tag{A.8}$$

Under the isomorphism (A.6), the Clifford action on χ is given by

$$(V^\Sigma \Gamma_\Sigma)\chi_\epsilon \sim i_x \chi + \xi \wedge \chi. \tag{A.9}$$

The spinor bilinear decomposes into the Mukai paring on the constituent forms

$$(\psi_\epsilon^t \cdot \chi_\epsilon)\epsilon = \langle \psi, \chi \rangle = \sum_P (-)^{[(p+1)/2]} \psi_p \wedge \chi_{6-p}. \tag{A.10}$$

The next subgroup one is interested in is the $O(6) \times O(6) \subset O(6,6)$ structure on E defined by a choice of metric g and B-field. Specifically in terms of the gamma matrices one can use g and B to change basis

$$\Gamma_m^\pm = \frac{1}{\sqrt{2}} \left(\check{\Gamma}_m + (B_{mn} \pm g_{mn})\hat{\Gamma}^n \right) \tag{A.11}$$

so the Clifford algebra becomes

$$\{\Gamma_m^+, \Gamma_n^-\} = 0, \qquad \{\Gamma_m^+, \Gamma_n^+\} = 2g_{mn}, \qquad \{\Gamma_m^-, \Gamma_n^-\} = -2g_{mn}. \tag{A.12}$$

In this basis $\mathcal{G}$ is block diagonal. Clearly $\Gamma^\pm$ generate two different $Spin(6)$ subgroups. We can correspondingly decompose the Clifford algebra $\text{Cliff}(6,6;\mathbb{R}) \simeq \text{Cliff}(6,0;\mathbb{R}) \times \text{Cliff}(6,0;\mathbb{R})$. The spinor bundle is then a product $S = S_1 \otimes S_2$ with $\chi_\epsilon = \theta_1 \otimes \theta_2$ and gamma matrices

$$\Gamma_m^+ = \gamma_m \otimes \mathbf{1}, \qquad \Gamma_m^- = \gamma_{(6)} \otimes \gamma_m, \tag{A.13}$$

where γ_m are defined above. The intertwiners $\mathcal{C}$ and $\tilde{\mathcal{D}}$ are given by

$$\mathcal{C} = C \otimes C\gamma_{(6)}, \qquad \tilde{\mathcal{D}} = D\gamma_{(6)} \otimes D\gamma_{(6)}. \tag{A.14}$$

The $O(6,6)$ chirality operator is given by

$$\Gamma_{(12)} = -\gamma_{(6)} \otimes \gamma_{(6)} \tag{A.15}$$

(and is manifestly independent of the sign of ϵ_g).

Finally, one can identify the common $O(6)$ subgroup of $GL(6,\mathbb{R})$ and $O(6) \times O(6)$. From this point of view θ_1 and θ_2 are spinors of the same $Spin(6)$ group and χ_ϵ is a bispinor. It is natural to represent χ_ϵ as

$$\chi_\epsilon = \tfrac{1}{4}\theta_1\theta_2^t(1 - \gamma_{(6)}) = \sum_p \frac{1}{8p!}\chi_{m_1...m_p}\gamma^{m_1...m_p}, \tag{A.16}$$

where the component forms are given by

$$\chi_{m_1...m_p} = \mathrm{tr}(\chi\gamma_{m_p...m_1}) \in \Lambda^p F . \tag{A.17}$$

The additional factor of $1 - \gamma_{(6)}$ is included so that the induced Clifford action on the forms χ_p is that given in (A.9). In terms of this representation (A.16) the spinor bilinear is given by

$$\psi_\epsilon^t \cdot \chi_\epsilon = -8\,\mathrm{tr}(\psi_\epsilon^t\chi_\epsilon) \tag{A.18}$$

where in this representation one has

$$\psi_\epsilon^t = \gamma_{(6)}C\psi_\epsilon^T C^{-1}, \tag{A.19}$$

which follows directly from (A.14) and (A.16). Similarly given the expression (A.14) for the intertwiner $\tilde{\mathcal{D}}$, we have

$$\chi_\epsilon^{\tilde{c}} = \tilde{\mathcal{D}}\chi^* = \gamma_{(6)}D\chi_\epsilon^* D^{-1}\gamma_{(6)}^{-1} . \tag{A.20}$$

In terms of the component forms $\chi_p^{\tilde{c}} = \chi_p^*$.

Let us finish by considering chiral spinors $\chi_\epsilon^\pm \in S^\pm$ in the representation (A.16). First we note that in this case the Clifford action can be written as

$$(V^\Sigma\Gamma_\Sigma)\chi_\epsilon^\pm = \tfrac{1}{2}[x^m\gamma_m, \chi_\epsilon^\pm]_\mp + \tfrac{1}{2}[\xi_m\gamma^m, \chi_\epsilon^\pm]_\pm. \tag{A.21}$$

Next, given the chirality operator (A.15), we see that real chiral spinors can be written as

$$\chi_\epsilon^\pm = \zeta_+\bar{\zeta}'_\pm \pm \zeta_-\bar{\zeta}'_\mp, \tag{A.22}$$

where $\zeta_\pm$ and $\zeta'_\pm$ are chiral $Spin(6)$ spinors. Note that as such they are eigenspinors of $1 - \gamma_{(6)}$ and comparing with (A.16) we see this form is compatible with $\zeta_\pm$ and $\zeta'_\pm$ being sections of the two spin bundles S_1 and S_2 respectively. Note that the sign between the two terms in (A.22) comes from the reality condition defined using (A.20).

In the main text we are interested in a pair of complex chiral $O(6,6)$ spinors given in the representation (A.16) by

$$\Phi_0^+ = \eta_+^1\bar{\eta}_+^2, \qquad \Phi_0^- = \eta_+^1\bar{\eta}_-^2. \tag{A.23}$$

Note, that, in this case we have

$$(\Phi_0^+)^{\tilde{c}} = \tilde{\mathcal{D}}(\Phi_0^+)^* = \eta_-^1\bar{\eta}_-^2, \qquad (\Phi_0^-)^{\tilde{c}} = \tilde{\mathcal{D}}(\Phi_0^-)^* = -\eta_-^1\bar{\eta}_+^2. \tag{A.24}$$

By a slight abuse of notation, in the main text we denote $(\Phi_0^\pm)^{\tilde{c}}$ by $\bar{\Phi}_0^\pm$. Note that we also have

$$(\Phi_0^+)^t = -\mathrm{i}\eta_-^2\bar{\eta}_-^1, \qquad (\Phi_0^-)^t = \mathrm{i}\eta_+^2\bar{\eta}_-^1. \tag{A.25}$$

B. Generic truncation

In this appendix we discuss the general conditions on mode truncations of the infinite tower of Kaluza-Klein states on M_6. In particular, we give the conditions such that there is a local special Kähler metric on the moduli space truncated modes, which is inherited from the local special Kähler geometry of the untruncated theory. A special case of such a truncation, is the expansion in terms of harmonic modes on a Calabi-Yau manifold.[19]

The section is divided as follows. We first recall the definition of (local) special Kähler geometry following the approach of [71]. We then review how this geometry is realised in the untruncated theory and finally derive the conditions for a special Kähler geometry on the truncated theory.

B.1 Special Kähler geometry

There are many different ways to define a rigid or local special Kähler geometry. One is as follows [71]. Let U be a $2d$-dimensional Kähler manifold with Kähler form ω and complex structure J. A rigid special Kähler structure on U is a flat torsion-free connection ∇ satisfying

$$\nabla_i \omega_{jk} = 0, \qquad \nabla_{[i} J^k{}_{j]} = 0. \tag{B.1}$$

The first condition is equivalent to the statement that one can find coordinates u^i whose transition functions are of the form

$$u^i = S^i{}_j u'^j + a^i, \tag{B.2}$$

where $S \in Sp(2d, \mathbb{R})$ is a constant symplectic transformation and $a \in \mathbb{R}^{2d}$. In these coordinates $\nabla_i = \partial_i$. The second condition means that locally one can introduce a vector $\hat{u} = \hat{u}^i \partial_i$ such that, in these coordinates,

$$J^i{}_j = -\partial_j \hat{u}^i. \tag{B.3}$$

Furthermore since the metric $g_{ij} = \omega_{ik} J^k{}_j$ is symmetric we have locally

$$\hat{u}^i = -(\omega^{-1})^{ij} \partial_j K \tag{B.4}$$

for some real function K. In addition, it is easy to see that K is actually the Kähler potential.

One can introduce special complex coordinates as follows. Given the coordinates u^i, locally one can define a vector field $u = u^i \partial_i$ and hence a local holomorphic vector field

$$\zeta = \tfrac{1}{2} (u + i\hat{u}). \tag{B.5}$$

From (B.2) and (B.3) we see that ζ is unique up to a shift by a constant complex vector. Furthermore

$$K_{\text{rigid}} = i\omega(\zeta, \bar{\zeta}). \tag{B.6}$$

---

[19] A discussion of the truncation conditions in the particular case of an $SU(3)$ structure also appeared very recently in [50] and appears to be in agreement with the analysis given here.

By making a symplectic transformation one can always choose Darboux coordinates $u^i = (x^I, y_I)$ with $I = 1, \ldots, d$ such that

$$\omega = \mathrm{d}x^I \wedge \mathrm{d}y_I. \tag{B.7}$$

In this basis one can write ζ as

$$\zeta = Z^I \frac{\partial}{\partial x^I} - \mathcal{F}_I \frac{\partial}{\partial y_I}. \tag{B.8}$$

The functions Z^I are special complex coordinates on the special Kähler manifold and the holomorphic functions $\mathcal{F}_I$ are locally given in terms of a prepotential $\mathcal{F}(Z)$, by $\mathcal{F}_I = \partial \mathcal{F}/\partial Z^I$.

A local special Kähler manifold can be viewed as a quotient of a rigid special Kähler manifold. Suppose U is a $2d + 2$ dimensional rigid special Kähler manifold such that one can find a globally defined holomorphic vector field ζ of the form (B.5) such that $\mathrm{Im}\,\zeta$ is a Killing vector field and the orbits of ζ define U as a $\mathbb{C}^*$ fibration over a base V. The space V is then a special Kähler manifold and the metric induced on V by taking the quotient by the $\mathbb{C}^*$ action is a local special Kähler metric. The special coordinates Z^I become projective special coordinates on V. The $\mathbb{C}^*$ symmetry implies that the prepotential $\mathcal{F}(Z)$ is homogeneous of degree two. The Kähler potential on V is given by

$$K = -\ln \mathrm{i}\omega(\zeta, \bar{\zeta}). \tag{B.9}$$

The moduli space of Calabi-Yau manifolds is a product of two special geometries spanned by the deformations of the Kähler form and the deformations of the complex structure [72].

B.2 Truncation conditions

The untruncated theory. Let us now review how special Kähler manifolds appear in the context of generalised geometry following [2–4] (see also [45] for a review). Let $S^\pm(E)$ be the positive and negative chirality generalised spinor bundles discussed in section 2 and $S_p^\pm(E)$ be the fibre at a point $p \in M^{9,1}$. One then considers an open subset $\mathcal{S}_p^\pm \subset S_p^\pm(E)$ of so-called stable spinors. These are the spinors with stabilizer group $SU(3,3)$. One finds that U is an open orbit under $O(6,6)$.

Hitchin then shows that there is a natural local special Kähler metric on $\mathcal{S}_p^\pm$. The construction is as follows. Since $S_p^\pm(E)$ is a vector space one can identify $T\mathcal{S}_p^\pm$ with $S_p^\pm(E)$ and define the symplectic structure ω in terms of the spinor bilinear (2.7), that is, for $\psi, \chi_\epsilon \in S_p^\pm(E) \simeq T\mathcal{S}_p^\pm$,

$$\omega(\psi_\epsilon, \chi_\epsilon) = \psi_\epsilon^t \cdot \chi_\epsilon. \tag{B.10}$$

One then chooses natural coordinates χ_ϵ^i which are just the components of the spinor $\chi_\epsilon \in \mathcal{S}_p^\pm$. Then by definition $\nabla_i \omega_{jk} = 0$ with $\nabla_i = \partial/\partial \chi_\epsilon^i$.

The complex structure is defined by the real function K_{rigid} via (B.3) and (B.4). On $\mathcal{S}_p^\pm$ it is given by Hitchin function

$$K_{\mathrm{rigid}} = H_\epsilon(\chi_\epsilon). \tag{B.11}$$

This is a particular $Spin(6,6)$ invariant homogeneous function of degree two. In the notation of [45] the holomorphic vector field ζ is given by

$$\Phi_\epsilon^\pm = \tfrac{1}{2}\left(\chi_\epsilon + \mathrm{i}\hat{\chi}_\epsilon\right) \tag{B.12}$$

where $\hat{\chi}_\epsilon^i = -(\omega^{-1})^{ij}(\partial H_\epsilon / \partial \chi_\epsilon^j)$, and is precisely the pure spinor Φ_ϵ^+ or Φ_ϵ^- discussed in section 2 which was used to define an $SU(3,3)$ structure.

Finally, the homogeneity of H_ϵ implies that $\hat{\chi}_\epsilon$ is a Killing vector field. Furthermore $\mathcal{S}_p^\pm$ is a $\mathbb{C}^*$ fibration, where $\Phi_\epsilon^\pm$ generates the $\mathbb{C}^*$ action on the fibres. This implies that the quotient $\mathcal{S}_p^\pm/\mathbb{C}^*$ is a local special Kähler manifold with Kähler potential

$$K = -\ln H_\epsilon. \tag{B.13}$$

Note that this implies that the corresponding metric is actually independent of the choice of volume form which defines the isomorphism between $S^\pm$ and $\Lambda^{\mathrm{even/odd}}$. These means that the how analysis could actually be repeated for stable forms $\chi \in \Lambda^{\mathrm{even/odd}}$. In this case, the symplectic structure gets replaced by the Mukai pairing (2.7) and the Hitchin function becomes a six-form

$$\mathrm{e}^{-K} = H = \mathrm{i}\left\langle \Phi^\pm, \bar{\Phi}^\pm \right\rangle \tag{B.14}$$

Crucially, the local special Kähler metric on V_p defined by (B.13) or (B.14), is related to the supergravity action. Specifically in the case of $SU(3)$ structure it was shown that the metrics on $\mathcal{S}_p^\pm/\mathbb{C}^*$ corresponding to the two pure spinors $\Phi^\pm$ are related to the corresponding kinetic terms in the rewriting of type II supergravity.

Defining the truncation. Now suppose that $M^{9,1} = M^{3,1} \times M^6$ so that $F = TM^6$. In analogy to keeping only the moduli of a Calabi-Yau manifold we would like to make a truncation, keeping some finite dimensional subspace of $SU(3,3)$ structures Φ on E. More formally let us start by defining a sub-bundle $\mathcal{S}^\pm \in S^\pm(E)$ of stable spinors (or the equivalent space of stable odd or even forms). The truncation is then an embedding map from some finite dimensional space U into the infinite dimensional space of sections $C^\infty(\mathcal{S}^\pm)$

$$\sigma : U \to C^\infty(\mathcal{S}^\pm). \tag{B.15}$$

In the case of a Calabi-Yau manifold, U is the odd or even cohomology and σ identifies harmonic forms with elements in U. For the truncation to be supersymmetric, we require that the special Kähler geometry on the fibres $\mathcal{S}_p^\pm$ induces a special Kähler metric on U. The purpose of this section is to find the constraints on the map σ such that this is true.

The first requirement is that U is a complex manifold. We have already seen that there is a natural complex structure on each fibre $\mathcal{S}_p^\pm$. Hence there is a complex structure $\mathcal{J}$ on $C^\infty(\mathcal{S}^\pm)$. This will descend to a complex structure on U if the embedding σ is holomorphic. Specifically, recall that σ induces the usual push-forward map $\sigma_* : TU \to T\mathcal{S}^\pm$ on vectors. We then define the complex structure J on U by requiring it to be compatible with the complex structure $\mathcal{J}$ on $C^\infty(\mathcal{S}^\pm)$, that is $\sigma_* J = \mathcal{J}\sigma_*$. Explicitly suppose u^i are coordinates on U. In general we can write the push-forward of a vector $t \in TU$ as

$$t = t^i\partial_i \mapsto \sigma_* t = t^i\Sigma_i(u) \tag{B.16}$$

where $\Sigma_i(u) = \partial_i \sigma$ form a basis[20] for the image of TU in $T\mathcal{S}^{\pm}$. In the special case of a Calabi-Yau manifold, $\Sigma_i(u)$ are harmonic forms. The complex structure J is then related to $\mathcal{J}$ by

$$\mathcal{J}\Sigma_i = J^j{}_i \Sigma_j. \tag{B.17}$$

In other words the image of Σ_i under $\mathcal{J}$ can still be expanded in the basis Σ_i. In the context of a Calabi-Yau manifold that action of $\mathcal{J}$ corresponds to taking the Hodge dual. The condition (B.17) then states that the Hodge dual of a harmonic form is itself harmonic.

We now turn to the symplectic structure on U. We have seen that the spinor bilinear (or equivalently the Mukai pairing) defines a symplectic structure on each fibre $\mathcal{S}_p$. We can define a bilinear on $C^\infty(T\mathcal{S})$ simply by integrating over M^6. Using σ_* we can then define a bilinear ω on TU by

$$\omega(s,t) = \int_{M^6} \langle \sigma_* s, \sigma_* t \rangle. \tag{B.18}$$

In components we have

$$\omega_{ij} = \int_{M^6} \langle \Sigma_i, \Sigma_j \rangle. \tag{B.19}$$

To be a symplectic structure we require that ω is non-degenerate. Using the Kähler structure on $\mathcal{S}_p$, it is then by construction compatible with J.

The next requirement is that (ω, J) is special Kähler. This means first that we can choose coordinate u^i such that $\partial_i \omega_{jk} = 0$ or equivalently

$$\int_{M^6} \langle \Sigma_j, \partial_i \Sigma_k \rangle = 0. \tag{B.20}$$

Again, the special Kähler structure on $\mathcal{S}_p$ then implies that $\partial_{[i} J^k{}_{j]} = 0$ and hence there is a rigid special Kähler metric on U.

Finally, of course, we actually want a local special Kähler metric, and hence some natural $\mathbb{C}^*$ action on U. Again, we have such an action on $\mathcal{S}_p$ generated by the holomorphic vector Φ and hence a $\mathbb{C}^*$ action on $C^\infty(\mathcal{S})$. Thus the natural requirement is that this induces a $\mathbb{C}^*$ action on U. In other words the holomorphic vector $\zeta \in TU$ of the form (B.5) which defines the rigid special Kähler structure on U satisfies $\sigma_* \zeta = \Phi$. This means that, on a coordinate patch u^i the map σ is realised by

$$u^i \mapsto u^i \Sigma_i. \tag{B.21}$$

Since we also have $\Sigma_i = \partial_i \sigma$ this requires that $u^i \partial_j \Sigma_i = 0$ or equivalently

$$u^i \partial_i \Sigma_j = 0, \tag{B.22}$$

that is, the basis forms Σ_i are homogeneous of degree zero. If this is satisfied, then there is a local special Kähler metric on $V = U/\mathbb{C}^*$. Furthermore, it is easy to show that the Kähler potential on V is given by

$$K = -\ln \int_{M^6} H = -\ln \mathrm{i} \int_{M_6} \langle \Phi^{\pm}, \bar{\Phi}^{\pm} \rangle \tag{B.23}$$

---

[20]In the main text, we use the notation $\Sigma_{\hat{A}}^+$ for the basis of even forms in $T\mathcal{S}^+$, and $\Sigma_{\hat{I}}^-$ for the basis of odd forms in $T\mathcal{S}^-$.

where H is the Hitchin function defined using the Mukai pairing.

Finally, it is convenient to rewrite these expressions in terms of Darboux coordinates $u^i = (x^I, y_I)$ with $I = 0, 1, \ldots, d$ such that $\omega = \mathrm{d}x^I \wedge \mathrm{d}y_I$. Distinguishing between the odd and even cases we have the corresponding bases

$$\Sigma^+ = \{\omega_A, \tilde{\omega}^B\}, \qquad \Sigma^- = \{\alpha_I, \beta^J\} \tag{B.24}$$

such that

$$\int_{M^6} \langle \alpha_I, \beta^J \rangle = \delta_I{}^J, \tag{B.25}$$

and $\int_{M_6} \langle \alpha_I, \alpha_J \rangle = \int_{M_6} \langle \beta^I, \beta^J \rangle = 0$, together with

$$\int_{M^6} \langle \omega_A, \tilde{\omega}^B \rangle = \delta_A{}^B, \tag{B.26}$$

and $\int_{M^6} \langle \omega_A, \omega_B \rangle = \int_{M^6} \langle \tilde{\omega}^A, \tilde{\omega}^B \rangle = 0$.

We can then introduce holomorphic coordinates Z^I (or X^A) and a prepotential $\mathcal{F}$ (or F) such that

$$\begin{aligned} \Phi^+ &= X^I \omega_A - F_A \tilde{\omega}^A, \\ \Phi^- &= Z^I \alpha_I - \mathcal{F}_I \beta^I. \end{aligned} \tag{B.27}$$

References

[1] For recents reviews see, for example:
M. Grana, *Flux compactifications in string theory: a comprehensive review*, *Phys. Rept.* **423** (2006) 91 [`hep-th/0509003`];
M.R. Douglas and S. Kachru, *Flux compactification*, `hep-th/0610102`;
R. Blumenhagen, B. Körs, D. Lüst and S. Stieberger, *Four-dimensional string compactifications with D-branes, orientifolds and fluxes*, `hep-th/0610327`, and references therein.

[2] N. Hitchin, *The geometry of three-forms in six and seven dimensions*, *J. Diff. Geom.* **55** (2000) 547 [`math.DG/0010054`].

[3] N. Hitchin, *Stable forms and special metrics*, in *Global Differential Geometry: The Mathematical Legacy of Alfred Gray*, M.Fernandez and J.A.Wolf (eds.), Contemporary Mathematics **288**, American Mathematical Society, Providence (2001) [`math.DG/0107101`].

[4] N. Hitchin, *Generalized Calabi-Yau manifolds*, *Quart. J. Math. Oxford Ser.* **54** (2003) 281 [`math.DG/0209099`].

[5] N. Hitchin, *Instantons, Poisson structures and generalized Kaehler geometry*, *Commun. Math. Phys.* **265** (2006) 131 [`math.dg/0503432`].

[6] N. Hitchin, *Brackets, forms and invariant functionals*, `math.dg/0508618`.

[7] S. Chiossi and S. Salamon, *The intrinsic torsion of $SU(3)$ and G_2 structures*, in *Differential geometry*, Valencia, (2001) [`math.DG/0202282`].

[8] M. Gualtieri, *Generalized complex geometry*, Oxford University, PhD thesis (2004) [`math.DG/0401221`].

[9] G.R. Cavalcanti, *New aspects of the DDC-lemma*, Oxford University, PhD thesis (2004) [math.dg/0501406].

[10] F. Witt, *Generalised g_2-manifolds*, *Commun. Math. Phys.* **265** (2006) 275 [math.dg/0411642]; *Special metric structures and closed forms*, Oxford University, PhD thesis (2004) [math.dg/0502443].

[11] S. Gurrieri, J. Louis, A. Micu and D. Waldram, *Mirror symmetry in generalized Calabi-Yau compactifications*, *Nucl. Phys.* **B 654** (2003) 61 [hep-th/0211102].

[12] S. Fidanza, R. Minasian and A. Tomasiello, *Mirror symmetric SU(3)-structure manifolds with NS fluxes*, *Commun. Math. Phys.* **254** (2005) 401 [hep-th/0311122].

[13] O. Ben-Bassat, *Mirror symmetry and generalized complex manifolds*, *J. Geom. Phys.* **56** (2006) 533 [math.ag/0405303].

[14] C. Jeschek, *Generalized Calabi-Yau structures and mirror symmetry*, hep-th/0406046.

[15] S. Chiantese, F. Gmeiner and C. Jeschek, *Mirror symmetry for topological sigma models with generalized Kaehler geometry*, *Int. J. Mod. Phys.* **A 21** (2006) 2377 [hep-th/0408169].

[16] A. Tomasiello, *Topological mirror symmetry with fluxes*, *JHEP* **06** (2005) 067 [hep-th/0502148].

[17] J. Louis and A. Micu, *Heterotic-type-IIA duality with fluxes*, *JHEP* **03** (2007) 026 [hep-th/0608171].

[18] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four-folds*, *Nucl. Phys.* **B 584** (2000) 69 [hep-th/9906070] [Erratum *ibid.* **608** (2001) 477]; S. Gukov, *Solitons, superpotentials and calibrations*, *Nucl. Phys.* **B 574** (2000) 169 [hep-th/9911011].

[19] J. Louis and A. Micu, *Type II theories compactified on Calabi-Yau threefolds in the presence of background fluxes*, *Nucl. Phys.* **B 635** (2002) 395 [hep-th/0202168].

[20] C. Vafa, *Superstrings and topological strings at large-N*, *J. Math. Phys.* **42** (2001) 2798 [hep-th/0008142].

[21] M. Rocek, *Modified Calabi-Yau manifolds with torsion*, in *Essays on Mirror Manifolds*, International Press, S.T. Yau ed., Hong Kong (1992); J. Gates, S. J., C.M. Hull and M. Roček, *Twisted multiplets and new supersymmetric nonlinear sigma models*, *Nucl. Phys.* **B 248** (1984) 157; C.M. Hull, *Compactifications of the heterotic superstring*, *Phys. Lett.* **B 178** (1986) 357; A. Strominger, *Superstrings with torsion*, *Nucl. Phys.* **B 274** (1986) 253.

[22] R. D'Auria, S. Ferrara, M. Trigiante and S. Vaula, *Gauging the Heisenberg algebra of special quaternionic manifolds*, *Phys. Lett.* **B 610** (2005) 147 [hep-th/0410290]; *Scalar potential for the gauged Heisenberg algebra and a non-polynomial antisymmetric tensor theory*, *Phys. Lett.* **B 610** (2005) 270 [hep-th/0412063].

[23] V. Mathai and J.M. Rosenberg, *On mysteriously missing T-duals, H-flux and the T-duality group*, hep-th/0409073; V. Mathai and J.M. Rosenberg, *T-duality for torus bundles via noncommutative topology*, *Commun. Math. Phys.* **253** (2004) 705 [hep-th/0401168]; P. Bouwknegt, K. Hannabuss and V. Mathai, *Nonassociative tori and applications to T-duality*, *Commun. Math. Phys.* **264** (2006) 41 [hep-th/0412092].

[24] C.M. Hull, *A geometry for non-geometric string backgrounds*, *JHEP* **10** (2005) 065 [`hep-th/0406102`].

[25] J. Shelton, W. Taylor and B. Wecht, *Nongeometric flux compactifications*, *JHEP* **10** (2005) 085 [`hep-th/0508133`].

[26] S. Hellerman, J. McGreevy and B. Williams, *Geometric constructions of nongeometric string theories*, *JHEP* **01** (2004) 024 [`hep-th/0208174`].

[27] A. Dabholkar and C. Hull, *Duality twists, orbifolds and fluxes*, *JHEP* **09** (2003) 054 [`hep-th/0210209`].

[28] S. Kachru, M.B. Schulz, P.K. Tripathy and S.P. Trivedi, *New supersymmetric string compactifications*, *JHEP* **03** (2003) 061 [`hep-th/0211182`].

[29] D.A. Lowe, H. Nastase and S. Ramgoolam, *Massive IIA string theory and matrix theory compactification*, *Nucl. Phys.* **B 667** (2003) 55 [`hep-th/0303173`].

[30] A. Flournoy, B. Wecht and B. Williams, *Constructing nongeometric vacua in string theory*, *Nucl. Phys.* **B 706** (2005) 127 [`hep-th/0404217`].

[31] A. Kapustin and Y. Li, *Topological sigma-models with H-flux and twisted generalized complex manifolds*, `hep-th/0407249`.

[32] G. Dall'Agata and S. Ferrara, *Gauged supergravity algebras from twisted tori compactifications with fluxes*, *Nucl. Phys.* **B 717** (2005) 223 [`hep-th/0502066`].

[33] C.M. Hull and R.A. Reid-Edwards, *Flux compactifications of string theory on twisted tori*, `hep-th/0503114`.

[34] J. Gray and E.J. Hackett-Jones, *On T-folds, G-structures and supersymmetry*, *JHEP* **05** (2006) 071 [`hep-th/0506092`].

[35] A. Dabholkar and C. Hull, *Generalised T-duality and non-geometric backgrounds*, *JHEP* **05** (2006) 009 [`hep-th/0512005`].

[36] A. Lawrence, M.B. Schulz and B. Wecht, *D-branes in nongeometric backgrounds*, *JHEP* **07** (2006) 038 [`hep-th/0602025`].

[37] C.M. Hull and R.A. Reid-Edwards, *Flux compactifications of M-theory on twisted tori*, *JHEP* **10** (2006) 086 [`hep-th/0603094`].

[38] C.M. Hull, *Global aspects of T-duality, gauged sigma models and T-folds*, `hep-th/0604178`.

[39] C.M. Hull, *Doubled geometry and T-folds*, `hep-th/0605149`.

[40] J. Shelton, W. Taylor and B. Wecht, *Generalized flux vacua*, *JHEP* **02** (2007) 095 [`hep-th/0607015`].

[41] I. Ellwood and A. Hashimoto, *Effective descriptions of branes on non-geometric tori*, *JHEP* **12** (2006) 025 [`hep-th/0607135`].

[42] P. Grange and S. Schafer-Nameki, *T-duality with H-flux: non-commutativity, T-folds and G x G structure*, `hep-th/0609084`.

[43] K. Becker, M. Becker, C. Vafa and J. Walcher, *Moduli stabilization in non-geometric backgrounds*, `hep-th/0611001`.

[44] I.T. Ellwood, *NS-NS fluxes in Hitchin's generalized geometry*, `hep-th/0612100`.

[45] M. Graña, J. Louis and D. Waldram, *Hitchin functionals in N = 2 supergravity*, *JHEP* **01** (2006) 008 [`hep-th/0505264`].

[46] C. Jeschek and F. Witt, *Generalised G_2-structures and type-IIB superstrings*, *JHEP* **03** (2005) 053 [`hep-th/0412280`].

[47] M. Grana, R. Minasian, M. Petrini and A. Tomasiello, *Generalized structures of N = 1 vacua*, *JHEP* **11** (2005) 020 [`hep-th/0505212`].

[48] I. Benmachiche and T.W. Grimm, *Generalized N = 1 orientifold compactifications and the Hitchin functionals*, *Nucl. Phys.* **B 748** (2006) 200 [`hep-th/0602241`].

[49] P. Grange and R. Minasian, *Modified pure spinors and mirror symmetry*, *Nucl. Phys.* **B 732** (2006) 366 [`hep-th/0412086`];
P. Koerber, *Stable D-branes, calibrations and generalized Calabi-Yau geometry*, *JHEP* **08** (2005) 099 [`hep-th/0506154`];
L. Martucci and P. Smyth, *Supersymmetric D-branes and calibrations on general N = 1 backgrounds*, *JHEP* **11** (2005) 048 [`hep-th/0507099`].

[50] A.-K. Kashani-Poor and R. Minasian, *Towards reduction of type-II theories on* SU(3) *structure manifolds*, *JHEP* **03** (2007) 109 [`hep-th/0611106`].

[51] W.-y. Chuang, S. Kachru and A. Tomasiello, *Complex/symplectic mirrors*, `hep-th/0510042`.

[52] J.P. Gauntlett, N.-w. Kim, D. Martelli and D. Waldram, *Fivebranes wrapped on slag three-cycles and related geometry*, *JHEP* **11** (2001) 018 [`hep-th/0110034`];
J.P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, *G-structures and wrapped NS5-branes*, *Commun. Math. Phys.* **247** (2004) 421 [`hep-th/0205050`];
J.P. Gauntlett, D. Martelli and D. Waldram, *Superstrings with intrinsic torsion*, *Phys. Rev.* **D 69** (2004) 086002 [`hep-th/0302158`].

[53] M. Gualtieri, *Generalized geometry and the Hodge decomposition*, `math.dg/0409093`.

[54] See, for example: J. Polchinski, *String theory*, **2**, Cambridge University Press (1998).

[55] L. Andrianopoli et al., *N = 2 supergravity and N = 2 super Yang-Mills theory on general scalar manifolds: symplectic covariance, gaugings and the momentum map*, *J. Geom. Phys.* **23** (1997) 111 [`hep-th/9605032`].

[56] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest and A. Van Proeyen, *New formulations of D = 10 supersymmetry and D8-O8 domain walls*, *Class. and Quant. Grav.* **18** (2001) 3359 [`hep-th/0103233`].

[57] J. Michelson, *Compactifications of type-IIB strings to four dimensions with non-trivial classical potential*, *Nucl. Phys.* **B 495** (1997) 127 [`hep-th/9610151`].

[58] T.R. Taylor and C. Vafa, *RR flux on Calabi-Yau and partial supersymmetry breaking*, *Phys. Lett.* **B 474** (2000) 130 [`hep-th/9912152`].

[59] G. Dall'Agata, *Type IIB supergravity compactified on a Calabi-Yau manifold with H-fluxes*, *JHEP* **11** (2001) 005 [`hep-th/0107264`].

[60] S. Gurrieri and A. Micu, *Type IIB theory on half-flat manifolds*, *Class. and Quant. Grav.* **20** (2003) 2181 [`hep-th/0212278`].

[61] G. Dall'Agata, R. D'Auria, L. Sommovigo and S. Vaula, *D = 4, N = 2 gauged supergravity in the presence of tensor multiplets*, *Nucl. Phys.* **B 682** (2004) 243 [`hep-th/0312210`];
L. Sommovigo and S. Vaula, *D = 4, N = 2 supergravity with abelian electric and magnetic charge*, *Phys. Lett.* **B 602** (2004) 130 [`hep-th/0407205`];
R. D'Auria, L. Sommovigo and S. Vaula, *N = 2 supergravity lagrangian coupled to tensor multiplets with electric and magnetic fluxes*, *JHEP* **11** (2004) 028 [`hep-th/0409097`].

[62] B. de Wit, H. Samtleben and M. Trigiante, *Magnetic charges in local field theory*, *JHEP* **09** (2005) 016 [`hep-th/0507289`].

[63] J. Louis and S. Vaula, *N = 1 domain wall solutions of massive type-II supergravity as generalized geometries*, *JHEP* **08** (2006) 058 [`hep-th/0605063`].

[64] P. Berglund and P. Mayr, *Non-perturbative superpotentials in F-theory and string duality*, `hep-th/0504058`.

[65] G. Dall'Agata, *Non-Kaehler attracting manifolds*, *JHEP* **04** (2006) 001 [`hep-th/0602045`].

[66] G. Aldazabal, P.G. Camara, A. Font and L.E. Ibáñez, *More dual fluxes and moduli fixing*, *JHEP* **05** (2006) 070 [`hep-th/0602089`].

[67] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, *Nucl. Phys.* **B 479** (1996) 243 [`hep-th/9606040`].

[68] N. Kaloper and R.C. Myers, *The O(dd) story of massive supergravity*, *JHEP* **05** (1999) 010 [`hep-th/9901045`].

[69] R. Minasian, M. Petrini and A. Zaffaroni, *Gravity duals to deformed SYM theories and generalized complex geometry*, *JHEP* **12** (2006) 055 [`hep-th/0606257`].

[70] M. Graña, R. Minasian, M. Petrini and A. Tomasiello, *A scan for new N = 1 vacua on twisted tori*, `hep-th/0609124`.

[71] D.S. Freed, *Special Kaehler manifolds*, *Commun. Math. Phys.* **203** (1999) 31 [`hep-th/9712042`].

[72] A. Strominger, *Yukawa couplings in superstring compactification*, *Phys. Rev. Lett.* **55** (1985) 2547; *Special geometry*, *Commun. Math. Phys.* **133** (1990) 163;
P. Candelas and X. de la Ossa, *Moduli space of Calabi-Yau manifolds*, *Nucl. Phys.* **B 355** (1991) 455.